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Ph.D. Thesis

# Kazhdan-Lusztig Theory: <br> Boolean Elements, Special Matchings and Combinatorial Invariance 

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## Introduction

Kazhdan-Lusztig theory lies in the intersection of different research areas of modern mathematics such as representation theory, algebraic geometry, Verma module theory, and combinatorics. In this thesis we tackle the subject from a combinatorial point of view, stressing its links with the combinatorics of words, the theory of posets, and the theory of matchings of posets.

Kazhdan-Lusztig theory originated in the paper [40] by D. Kazhdan and G. Lusztig of 1979. In this seminal paper, the authors introduced a new family of representations of the Hecke algebra, which is a sort of deformation of the group algebra of the Coxeter group. The Hecke algebra and its representations relate to two families of polynomials with integer coefficients, indexed by pairs of elements in the Coxeter group, now commonly referred to as the family $\left\{R_{u, v}(q)\right\}_{u, v \in W}$ of $R$-polynomials and the family $\left\{P_{u, v}(q)\right\}_{u, v \in W}$ of Kazhdan-Lusztig polynomials. These two families are strictly connected together (and are actually in some sense equivalent), and are related to the Bruhat order of the underlying Coxeter group.

After [40], which has become a turning point in Coxeter group theory, a large number of mathematicians started studying these subjects and their related topics. Kazhdan-Lusztig polynomials have been proven to have several applications in different contexts. We do not want here to make a list of these applications and we refer the interested reader to [3], [29], [36], [40], [41]. We just want to briefly recall the two following connections with Hecke algebras and Schubert varieties, which are of concern to us. The Kazhdan-Lusztig representations of the Hecke algebra introduced in [40] are based on certain graphs (called $W$-graphs in [40]). The main ingredients for the construction of certain $W$-graphs are the top coefficients of the Kazhdan-Lusztig polynomials of the group. This is the main reason why the function $\mu$ is important. As to the role of Kazhdan-Lusztig polynomials in the geometry of Schubert varieties, it is
known that, for Weyl and affine Weyl groups, their coefficients are a measure of the singularities of the corresponding Schubert varieties. They actually count the dimensions of the local intersection homology spaces of these varieties at a point lying in a given Schubert cell.

Once these applications of Kazhdan-Lusztig polynomials had been found, there followed the problem of computing them. The main tools are fairly complicated recursive formulae already appearing in [40]. In the past twenty years, many mathematicians have tried to deduce non recursive closed formulae, at least for small classes of elements in particular Coxeter groups (mainly in the symmetric group). For explicit descriptions of some families of Kazhdan-Lusztig polynomials we refer to the works of Billey and Warrington [4], Brenti and Simion [19], Boe [8], Lascoux and Schützenberger [45], Shapiro, Shapiro and Vainshtein [53].

The recurrence satisfied by the Kazhdan-Lusztig polynomial $P_{u, v}(q)$ depends on the descents of $u$ and $v$, on the Kazhdan-Lusztig polynomials $P_{x, y}(q)$ for all $x, y$ in the interval $[u, v]$, and on $[u, v]$ as a partially order set under the Bruhat order. One of the most famous conjectures of Kazhdan-Lusztig theory is due to Lusztig and states that the Kazhdan-Lusztig polynomial $P_{u, v}(q)$ actually depends only on the isomorphism type of the interval $[u, v]$ as a poset. As customary, we refer to this conjecture as the conjecture of the combinatorial invariance of Kazhdan-Lusztig polynomials. In a very recent paper [17], Brenti has proved the combinatorial invariance of Kazhdan-Lusztig polynomials in the case of the symmetric group $\mathfrak{S}(n)$ for lower Bruhat intervals. More precisely, he has proved that the Kazhdan-Lusztig polynomial indexed by the permutations $u$ and $v$ actually depends only on the isomorphism type of the interval $[e, v]$, where $e$ is the identity element of $\mathfrak{S}(n)$.

This thesis contains most of the results I have obtained in Kazhdan-Lusztig theory under the accurate and always encouraging direction of Prof. F. Brenti. It is divided into two distinct parts.

The first part, comprising Chapters 1-3, is the result of the work I have done after having proved a conjectures by Brenti regarding certain explicit formulae for $R$-polynomials of the symmetric group. I realized that this proof works in a more general setting and the Boolean elements naturally came out (for the definition, see Section 1.1). Hence I tried to develop the theory for this class of elements with particular regard to explicit closed formulae. In particular,
here I compute the $R$-polynomials of any Coxeter group, the Kazhdan-Lusztig polynomials of a linear Coxeter group (see Section 0.4 for the definition), and the parabolic Kazhdan-Lusztig and $R$-polynomials of the symmetric group. All this formulae are easily stated in terms of certain tableaux associated to pairs of Boolean elements.
These formulae, moreover, turn out to have several consequences. They allow us to explicitly list all the pairs $(u, v)$ of Boolean elements with $\mu(u, v) \neq 0$, to compute and factorize the Kazhdan-Lusztig elements indexed by Boolean elements, to compute and factorize the intersection homology Poincaré polynomials indexed by Boolean elements, to prove Lusztig's conjecture of the combinatorial invariance for Boolean elements. In all these results, $(W, S)$ can be any linear Coxeter system except in the last one, where ( $W, S$ ) is supposed to be strictly linear.

The second part, comprising Chapters 5-7, is the result of a pleasant and fruitful collaboration with Francesco Brenti and Fabrizio Caselli, which is still ongoing. This cooperation started while trying to give a solution to Lusztig's conjecture on the combinatorial invariance of Kazhdan-Lusztig polynomials. The main result of Part II is certainly the following, which prove Lusztig's conjecture for lower Bruhat intervals in any Coxeter system.

Theorem. Let $(W, S)$ and ( $W^{\prime}, S^{\prime}$ ) be two Coxeter systems, $w \in W, w^{\prime} \in W^{\prime}$, and let $e$ and $e^{\prime}$ be the identities of $W$ and $W^{\prime}$, respectively. Suppose that $\phi:[e, w] \rightarrow\left[e^{\prime}, w^{\prime}\right]$ is an isomorphism of partially ordered sets (under the Bruhat order). Then, for all $u, v \in W, u, v \leq w$, the Kazhdan-Lusztig polynomial $P_{u, v}$ is equal to the Kazhdan-Lusztig polynomial $P_{\phi(u), \phi(v)}$.

The proof of this theorem uses the fundamental concept of special matchings of a partially ordered set, which are, by definition, combinatorial invariant. The crucial point is to prove that any special matching of $[e, v]$ leads to a poset theoretical way for computing the Kazhdan-Lusztig polynomials $P_{u, v}$ for all elements $u \leq v$. This result has many consequences. In particular we show several combinatorial formulae for both $R$-polynomials and Kazhdan-Lusztig polynomials which depend on classical combinatorial objects such as sub-sequences, paths in a label graph, compositions and lattice paths. This is done by introducing three families of sequences of special matchings which are all new combinatorial analogues of the concept of reduced expression.

The following is the plan of this thesis.
Part I is organized around the class of Boolean elements.
In Chapter 1 we introduce the Boolean elements and we give the preliminary lemmas that make the combinatorics of these elements easier.
In Chapter 2, we study the Kazhdan-Lusztig and $R$-polynomials indexed by Boolean elements. In particular, in Section 1 and in Section 2, we give closed product formulae for the $R$-polynomials of any Coxeter group and for the Kazhdan-Lusztig polynomials of any linear Coxeter group. As a consequence of these formulae, in Section 3 we prove Lusztig's conjecture of the combinatorial invariance for Boolean elements in strictly linear Coxeter systems. In Section 4, we explicitly list all the pairs $(u, v)$ of Boolean elements with $\mu(u, v) \neq 0$. This result can be useful also for the computation of other classes of Kazhdan-Lusztig polynomials since the function $\mu$ is often the main obstacle in their recursive computation (see, for example, [23, 24]). In Section 5 and in Section 6 we compute and factorize respectively the Kazhdan-Lusztig elements and the intersection homology Poincaré polynomials indexed by Boolean elements.

In Chapter 3, we compute the parabolic analogues of the Kazhdan-Lusztig and $R$-polynomials for the symmetric group in the case when the indexing permutations are Boolean. These formulae are valid with no restrictions on the parabolic subgroup $W_{J}$ and depend on the number of occurrences of certain sub-tableaux in a fixed tableau associated to the indexing permutations.

Part II is organized around the applications of the concept of special matching in Kazhdan-Lusztig theory.

Chapter 4 is devoted to the proof of Lusztig's conjecture on the combinatorial invariance of Kazhdan-Lusztig polynomials for lower intervals, that is for intervals of the form $[e, v]$ for any element $v$ in any Coxeter group. We start by giving some combinatorial properties of Bruhat order in Section 1 and by examining the combinatorics of pairs of special matchings in Section 2. After this, we tackle the problem of the combinatorial invariance. First, in Section 3, we prove the conjecture for lower Bruhat intervals in Coxeter groups of rank 3 and then from this, in Section 4, we deduce the result for all Coxeter groups. This follows by proving that special matchings lead to a poset theoretic recursion for computing $R$-polynomials (Corollary 4.4.8). Finally, in Section 5, for each $v \in W$, we introduce and study a combinatorial version of the Hecke algebra naturally associated to the special matchings of $[e, v]$ and an action of it on the submodule of the classical Hecke algebra of $W$ spanned by $\left\{T_{u}: u \leq v\right\}$. This
action enables us to reformulate Corollary 4.4 .8 in a very compact way by saying that this action "respects" the canonical involutions $\iota$ of these Hecke algebras. This, in turn, implies that the usual recursion for Kazhdan-Lusztig polynomials holds also when descents are replaced by special matchings thus giving a poset theoretic recursion for the Kazhdan-Lusztig polynomials which does not involve the $R$-polynomials.
In Chapter 5, we introduce three families of sequences of special matchings: the regular sequences, the $B$-regular sequences, and the $R$-regular sequences. All of them are new combinatorial analogues of the concept of reduced expression. Using these sequences, we generalize some formulae valid for Kazhdan-Lusztig and $R$-polynomials of any Coxeter system. In particular, in Section 1 we generalize an algorithm and a closed formula of Deodhar ([28, Algorithm 4.11] and [26, Theorem 1.3]) for Kazhdan-Lusztig and $R$-polynomials, respectively. In Section 2 we obtain a bijection between subsequences of $B$-regular sequences and certain paths in an appropriate directed graph. This bijection has several nice properties, and transforms the concepts and statistics used in the previous section into familiar ones on paths. In Section 3 we generalize to a combinatorially invariant setting what is probably the most explicit non-recursive formula known for Kazhdan-Lusztig polynomials which holds in complete generality, namely Theorem 7.3 of [14].
In Chapter 6, we study the set of all special matchings $S_{v}$ of a permutation $v$. We show that the group $\widehat{W}_{v}$ generated by the special matchings of $S_{v}$, which are involutions, is actually a Coxeter group, with $S_{v}$ as set of Coxeter generators. The Coxeter system $\left(\widehat{W}_{v}, S_{v}\right)$ is always isomorphic to a direct product of symmetric groups.
Finally, Chapter 7 deals with the problem of generalizing the definition of Kazhdan-Lusztig and $R$-polynomials to arbitrary posets. We prove that, in a certain class of posets, the concept of special matching leads to an entirely poset theoretic definition of Kazhdan-Lusztig and $R$-polynomials. This class of posets, which we call diamonds, includes the lower Bruhat intervals and the new definitions are obviously consistent with the classical definitions.
Chapter 0 is not meant to be an introduction either to Coxeter group theory or to Kazhdan-Lusztig theory. It just reviews the background material that is being used in both Part I and Part II, and collects some already known results for later reference. Rarely, some external references were necessary in Part II, but we have tried to minimize reliance on other sources. We refer to [39] and [9] for a detailed treatment of the subject.

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## Chapter 0

## Notation and Background

This chapter reviews the background material on posets, Coxeter systems and Kazhdan-Lusztig theory that is needed in the rest of this work.

### 0.1 Notation

We collect here some notation that will be adhered to in the sequel.

```
    \mathbb{Z}}=\mathrm{ ring of integer;
    P}=\mathrm{ set of positive integer;
    N}=\mathrm{ set of non-negative integer;
    Q = field of rational numbers;
    C}=\mathrm{ field of complex numbers;
    |S| = cardinality of S, for any set S;
[a,b]={n\in\mathbb{P}:a\leqn\leqb}, for }a,b\in\mathbb{N}
    [n] = [1,n], for }n\in\mathbb{N}
R[q] = ring of polynomials with coefficients in R for R=\mathbb{N},\mathbb{Z},\mathbb{Q},\mathbb{C};
[q}\mp@subsup{q}{}{i}]P=\mathrm{ the coefficient of }\mp@subsup{q}{}{i}\mathrm{ in }P\quad\mathrm{ for }i\in\mathbb{N},P\inR[q]
```

We write ":=" if we are defining the left hand side by the right hand side.

For a proposition $P$ we let

$$
\chi(P):= \begin{cases}1, & \text { if } P \text { is true } \\ 0, & \text { otherwise }\end{cases}
$$

If $a_{1}, \ldots, a_{k} \in \mathbb{Z}$, we write $S=\left\{a_{1}, \ldots, a_{k}\right\}_{<}$to mean that $S=\left\{a_{1}, \ldots, a_{k}\right\}$ and $a_{1}<\cdots<a_{k}$.

For $n \in \mathbb{P}$, we denote by $\mathfrak{S}(n)$ the group of all bijections $\pi:[n] \rightarrow[n]$ (the symmetric group). If $\sigma \in \mathfrak{S}(n)$ then we write $\sigma=\sigma_{1} \ldots \sigma_{n}$ to mean that $\sigma(i)=\sigma_{i}$, for $i=1, \ldots, n$. We will also write $\sigma$ in disjoint cycle form (see, e.g., [55], p.17) and we will usually omit writing the 1-cycles of $\sigma$. For example, if $\sigma=365492187$ then we also write $\sigma=(9,7,1,3,5)(2,6)$. Given $\sigma, \tau \in \mathfrak{S}(n)$ we let $\sigma \tau=\sigma \circ \tau$ (composition of functions) so that, for example, $(1,2)(2,3)=(1,2,3)$.

### 0.2 Posets

A partially ordered set $(P, \leq)$, or poset for short, consists of a set $P$ together with a partial order relation " $\leq$ ". The relation is suppressed from the notation when it is clear from context. A subset $R$ of $P$ has a structure of a poset with the order relation induced by $P$. An element $x \in P$ is maximal (respectively minimal) if there is no element $y \in P \backslash\{x\}$ such that $x \leq y$ (respectively $y \leq x$ ). We say that $P$ has a bottom element $\widehat{0}$ if there exists an element $\widehat{0} \in P$ satisfying $\widehat{0} \leq x$ for all $x \in P$. Similarly, $P$ has a top element $\hat{1}$ if there exists an element $\widehat{1} \in P$ satisfying $x \leq \widehat{1}$ for all $x \in P$. If both $\widehat{0}$ and $\hat{1}$ exist, then $P$ is bounded.

Two elements $x, y \in P$ are said to be comparable if either $x \leq y$ or $y \leq x$, and incomparable otherwise. We say that $P$ is connected if there do not exist two non-void subsets of $P$ such that any element of the first is incomparable with any element of the second. We also write $y \geq x$ to mean $x \leq y, x<y$ to mean $x \leq y$ and $x \neq y$, and $y>x$ to mean $x<y$. If $x \leq y$ we define the (closed) interval $[x, y]=\{z \in P: x \leq z \leq y\}$ and the open interval $(x, y)=\{z \in P: x<z<y\}$. If every interval of $P$ is finite, then $P$ is called a locally finite poset. We say that $y$ covers $x$, or $x$ is covered by $y$, if $x<y$ and $[x, y]=\{x, y\}$, and we write $x \triangleleft y$ as well as $y \triangleright x$. If $P$ has a $\hat{0}$ then an element $x \in P$ is an atom of $P$ if $\hat{0} \triangleleft x$. Similarly, if $P$ has a $\hat{1}$ then an element $x \in P$ is a coatom of $P$ if $x \triangleleft \hat{1}$.

The standard way of depicting a finite poset $P$ is to draw its Hasse diagram. This is the graph with $P$ as node set and having an upward-directed edge from $x$ to $y$ if and only if $x \triangleleft y$ (so $y$ is drawn "above" $x$ ). The Hasse diagram gives all the order relations by transitivity and it is clearly minimal with this property.

A sequence $C=\left(x_{0}, x_{1}, \ldots, x_{h}\right)$ of elements in $P$ is called a chain (respectively multichain) if $x_{0}<x_{1}<\ldots<x_{h}$ (respectively, $x_{0} \leq x_{1} \leq \ldots \leq x_{h}$ ). We then also say that $C$ starts with $x_{0}$ and ends with $x_{h}$. The integer $h$ is the length of $C$ and it is denoted by $l(C)$. The length of a finite poset $P$ is $l(P):=\max \{l(C): C$ is a chain of $P\}$. A chain is maximal if its elements are not a proper subset of those of any other chain. A chain is saturated if all successive relations are coverings: in this case we write $x_{0} \triangleleft x_{1} \triangleleft \cdots \triangleleft x_{h}$.

A morphism of posets is a map $\phi: P \rightarrow Q$ from the poset $P$ to the poset $Q$ which is order-preserving, namely such that $x \leq y$ in $P$ implies $\phi(x) \leq \phi(y)$ in $Q$, for all $x, y \in P$. If instead $x \leq y$ implies $\phi(x) \geq \phi(y)$, the map is orderreversing. Two posets $P$ and $Q$ are isomorphic if there exists an order-preserving bijection $\phi: P \rightarrow Q$ whose inverse is also order-preserving. In this case $\phi$ is an isomorphism of posets. An isomorphism of posets $\phi: P \rightarrow P$ is also called an automorphism. If, instead, $\phi: P \rightarrow P$ is a bijection such that $\phi$ and $\phi^{-1}$ are order-reversing, then $\phi$ is called an anti-automorphism. A poset $P$ is a Boolean algebra if there is a set $S$ such that $P$ is isomorphic to the set of all subsets of $S$, partially ordered by inclusion.

A poset $P$ is ranked if there exists a (rank) function $\rho: P \rightarrow \mathbb{N}$ such that $\rho(y)=\rho(x)+1$ whenever $x \triangleleft y$. A poset $P$ is pure of length $n$ if all maximal chains are of the same length $n$. A poset $P$ with bottom element $\widehat{0}$ is graded if every interval $[\widehat{0}, x], x \in P$, is pure. Suppose that $P$ is either pure or graded. Define the rank $\rho(x)$ of $x \in P$ to be the length of the subposet $\{y \in P: y \leq x\}$. This gives $P$ a structure of ranked poset.

The Möbius function of $P$ assigns to each ordered pair $x \leq y$ an integer $\mu(x, y)$ according to the following recursion:

$$
\mu(x, y)= \begin{cases}1, & \text { if } x=y  \tag{1}\\ -\sum_{x \leq z<y} \mu(x, z), & \text { if } x<y\end{cases}
$$

We say that a finite graded bounded poset $P$, with rank function $\rho$, is Eulerian if $\mu(u, v)=(-1)^{\rho(v)-\rho(u)}$ for all $u, v \in P, u \leq v$. Equivalently, $P$ is Eulerian if
and only if

$$
\mid\{p \in[u, v]: \rho(p) \text { is even }\}|=|\{p \in[u, v]: \rho(p) \text { is odd }\} \mid
$$

for all $u, v \in P, u \leq v$.
Let $\operatorname{Int}(P):=\left\{(x, y) \in P^{2}: x \leq y\right\}$. Given a commutative ring $R$, the incidence algebra $I(P ; R)$ of $P$ with coefficients in $R$ is the set of all functions $f: \operatorname{Int}(P) \rightarrow R$ with sum and product defined by

$$
(f+g)(x, y):=f(x, y)+g(x, y)
$$

and

$$
\begin{equation*}
(f g)(x, y):=\sum_{x \leq z \leq y} f(x, z) g(z, y) \tag{2}
\end{equation*}
$$

for all $f, g \in I(P ; R)$ and $(x, y) \in \operatorname{Int}(P)$. The incident algebra $I(P ; R)$ is an associative algebra having, as identity, the function $\delta$ defined by

$$
\delta(x, y):=\left\{\begin{array}{cc}
1 & \text { if } x=y \\
0 & \text { otherwise }
\end{array}\right.
$$

An element $f \in I(P ; R)$ is invertible if and only if $f(x, x)$ is invertible for all $x \in P$. If $f$ is invertible then we denote by $f^{-1}$ its (two-sided) inverse.

### 0.3 Coxeter systems

Let $S=\left\{s_{1}, \ldots, s_{r}\right\}$ be a finite set of cardinality $r$. A Coxeter matrix is a matrix $m: S \times S \rightarrow\{1,2, \ldots, \infty\}$ such that

1. $m\left(s_{i}, s_{j}\right)=m\left(s_{j}, s_{i}\right)$;
2. $m\left(s_{i}, s_{j}\right)=1 \Longleftrightarrow i=j$.
for all $i, j \in[r]$.
Any Coxeter matrix uniquely determines a group $W$ given by the presentation:

- generators: $S$;
- relations: $\left(s_{i} s_{j}\right)^{m\left(s_{i}, s_{j}\right)}$ for all $i, j \in[r]$ with $m\left(s_{i}, s_{j}\right) \neq \infty$.

If a group $W$ has such a presentation, then $W$ is a Coxeter group, the pair $(W, S)$ is a Coxeter system, and $S$ is a set of Coxeter generators. The cardinality
$|S|=r$ of $S$ is usually called the rank of $W$. Given two Coxeter systems ( $W, S$ ) and $\left(W^{\prime}, S^{\prime}\right)$, a map $\Phi: W \rightarrow W^{\prime}$ is an isomorphism of Coxeter systems if it is an isomorphism of groups and $\Phi(S)=S^{\prime}$. The isomorphism type of a Coxeter system $(W, S)$ is not determined by the isomorphism type of the group $W$ alone. Nevertheless, it is very common to talk about Coxeter groups while having in mind Coxeter systems.

The Coxeter matrix $m$ of a Coxeter system $(W, S)$ is encoded in its Coxeter graph. This is the labeled graph obtained in the following way: take $S=$ $\left\{s_{1}, \ldots, s_{r}\right\}$ as the set of vertices, then join a pair of vertices $\left\{s_{i}, s_{j}\right\}$ by an edge if and only if $m\left(s_{i}, s_{j}\right) \geq 3$ and label such an edge by $m\left(s_{i}, s_{j}\right)$ (labels equal to 3 are usually omitted).

By property 2 of the definition of Coxeter matrix, all generators are involutions. Hence any element $w \in W$ can be written as a product of generators (without using inverses)

$$
w=s_{i_{1}} \cdots s_{i_{t}}, \quad s_{i_{j}} \in S
$$

If $t$ is minimal among all such expression of $w$, then $t$ is the length of $w$ and it is denoted by $l(w)$. Any expression of $w$ which is a product of $l(w)$ elements of $S$ is called a reduced expression of $w$. There is only one element of length zero, the identity, which we denote by $e$.
For all $u, v \in W$, we let

$$
\begin{aligned}
D_{L}(u) & :=\{s \in S: l(s u)<l(u)\} \\
D_{R}(u) & :=\{s \in S: l(u s)<l(u)\} \\
T(W) & :=\left\{w s w^{-1}: s \in S, w \in W\right\}, \quad \text { (the set of reflections of } W \text { ). }
\end{aligned}
$$

The elements of $S$ are also called simple reflections. We write only $T$ instead of $T(W)$ when no confusion arises.

The proof of the following fundamental result can be found in [39] §5.8.

Theorem 0.3.1 (Exchange Property) Let $w \in W, s_{1}, s_{2}, \ldots, s_{r} \in S$, $w=$ $s_{1} s_{2} \ldots s_{r}$ where this expression is reduced. Let $t \in T(W)$ be such that $l(w t)<$ $l(w)$. Then there exists a unique $i \in[r]$ such that $w t=s_{1} s_{2} \ldots \widehat{s_{i}} \ldots s_{r}$ (where $\widehat{s_{i}}$ means that $s_{i}$ has been omitted). In particular, if $t \in S$, this $i \in[r]$ is such that $s_{i+1} s_{i+2} \ldots s_{r} s$ is reduced while $s_{i} s_{i+1} \ldots s_{r} s$ is not.

For the reader's convenience, we just record the following easy consequence of the Exchange Property.

Proposition 0.3.2 Given a Coxeter system $(W, S)$, let $u \in W$. If $s \in D_{L}(u)$, then there exists a reduced expression $s_{1} \cdots s_{r}$ of $u$ such that $s_{1}=s$. Dually, if $s \in D_{R}(u)$, then there exists a reduced expression $s_{1} \cdots s_{r}$ of $u$ such that $s_{r}=s$.

We will always assume that $W$ is partially ordered by (strong) Bruhat order (denoted by $\leq$ ), that we define through the following Theorem-Definition. By a subword of a word $s_{1} s_{2} \cdots s_{n}$ we mean a word of the form $s_{i_{1}} s_{i_{2}} \cdots s_{i_{r}}$, where $1 \leq i_{1}<1_{2}<\cdots<i_{r} \leq n$.

Theorem 0.3.3 Let $u, v \in W$. Then the following are equivalent:

1. $u \leq v$ in the Bruhat order;
2. there exist $t_{1}, \ldots, t_{r} \in T(W)$ such that $t_{r} \ldots t_{1} u=v$ and $l\left(t_{i} \ldots t_{1} u\right)>$ $l\left(t_{i-1} \ldots t_{1} u\right)$ for $i=1, \ldots, r$;
3. there exist $t_{1}, \ldots, t_{r} \in T(W)$ such that $u t_{1} \ldots t_{r}=v$ and $l\left(u t_{1} \ldots t_{i}\right)>$ $l\left(u t_{1} \ldots t_{i-1}\right)$ for $i=1, \ldots, r$;
4. for any reduced expression of $v$ there exists a reduced expression of $u$ which is a subword of it;
5. for every reduced expression of $v$ there exists a reduced expression of $u$ which is a subword of it.

The Bruhat order gives $W$ the structure of a graded poset, with length as rank function. If $u \leq v$ we let $l(u, v):=l(v)-l(u)$. As for every ranked poset, we write $u \triangleleft v$ if $u \leq v$ and $l(u, v)=1$. Given $u, v \in W$ we let $[u, v]_{W}:=\{x \in W$ : $u \leq x \leq v\}$ and we write $[u, v]$ when no confusion arises. We consider $[u, v]$ as a poset with the partial ordering induced by $W$. It is well known (see, e.g., [6], Corollary 1) that intervals of $W$ (and their duals) are Eulerian posets. Hence, in particular, if $l(u, v) \leq 2$ then all intervals $[u, v]$ have cardinality equal to 4 .

The Bruhat graph of $W$ is the following directed graph. Take $W$ as vertex set. For $u, v \in W$, put an arrow $u \rightarrow v$ from $u$ to $v$ if and only if $l(u)<l(v)$ and $u t=v$ (equivalently $t u=v$ ) for some reflection $t$. Clearly $u<v$ if and only if there exists a chain $u \rightarrow u_{1} \rightarrow u_{2} \rightarrow \cdots \rightarrow u_{k}=v$.

The following Lemma is usually referred to as the Lifting Lemma (see [39], Lemma 7.4 for a proof).

Lemma 0.3.4 (Lifting Lemma) Let $s \in S$ and $u, v \in W, u \leq v$. Then

1. if $s \in D_{R}(v)$ and $s \in D_{R}(u)$ then $u s \leq v s$;
2. if $s \notin D_{R}(v)$ and $s \notin D_{R}(u)$ then $u s \leq v s$;
3. if $s \in D_{R}(v)$ and $s \notin D_{R}(u)$ then $u s \leq v$ and $u \leq v s$.

We now recall some results due to J. Tits [60]. Given $s, t \in S$ such that $m(s, t)<\infty$, let $\alpha_{s, t}=\underbrace{s t s t \ldots}_{m(s, t)}$, with exactly $m(s, t)$ letters.

Lemma 0.3.5 Let $w \in W$ and $s, t \in D_{L}(w)$. Then there exists a reduced expression of $w$ which starts with $\alpha_{s, t}$, that is

$$
v=\alpha_{s, t} v^{\prime}
$$

with $l(v)=m(s, t)+l\left(v^{\prime}\right)$.
Dually, if $s, t \in D_{R}(w)$, then there exists a reduced expression of $w$ which ends with $\alpha_{s, t}$.

Two expressions are said to be linked by a braid move (respectively a nil move) if it is possible to obtain the first from the second by changing a factor $\alpha_{s, t}$ to a factor $\alpha_{t, s}$ (respectively by deleting a factor $s s$ ).

Theorem 0.3.6 (Tits' Word Theorem) Let $u \in W$. Then:

1. any two reduced expressions of $u$ are linked by a finite sequence of braid moves;
2. any expression of $u$ (not necessarily reduced) is linked to any reduced expression of $u$ by a finite sequence of braid and nil moves.

Let $J \subseteq S$. The subgroup of $W$ generated by the set $J$ is called the parabolic subgroup generated by $J$, and it is denoted by $W_{J}$. The pair $\left(W_{J}, J\right)$ itself is a Coxeter system with the relations induced by $(W, S)$. We denote by $W^{J}$ the set of minimal length representatives for the right cosets:

$$
W^{J}=\left\{w \in W: D_{L}(w) \subseteq S \backslash J\right\}
$$

We have the following decomposition.
Theorem 0.3.7 Multiplication gives a bijection $W_{J} \times W^{J} \rightarrow W$. That is, for all $w \in W$, there exist unique $w_{J} \in W_{J}$ and $w^{J} \in W^{J}$ such that

$$
w=w_{J} w^{J}
$$

Furthermore, these elements satisfy

$$
l(w)=l\left(w_{J}\right)+l\left(w^{J}\right)
$$

Note that $W^{\emptyset}=W$. If $W_{J}$ is finite then we denote by $w_{0}^{J}$ its longest element. Given $u, v \in W^{J}$, we let

$$
[u, v]_{J}=\left\{z \in W^{J}: u \leq z \leq v\right\}
$$

and consider $W^{J}$ and $[u, v]_{J}$ as posets with the partial ordering induced by $W$.

We refer the reader to [9] or to [39] for a more detailed treatment of the argument.

### 0.4 Symmetric groups and linear Coxeter groups

The most important Coxeter group is certainly the symmetric group $\mathfrak{S}(n)$, that is the group of all permutations of the set [ $n$ ].
Consider a set $S$ of cardinality $n-1$, say $S=\left\{s_{1}, s_{2}, \ldots, s_{n-1}\right\}$, and consider the Coxeter matrix $m$ given by:

$$
m\left(s_{i}, s_{j}\right)= \begin{cases}1, & \text { if }|i-j|=0 \\ 3, & \text { if }|i-j|=1 \\ 2, & \text { if }|i-j|>1\end{cases}
$$

for all $i, j \in[n-1]$. Call $W$ the Coxeter group associated to the Coxeter matrix $m$. We obtain a group isomorphism from $W$ to $\mathfrak{S}(n)$ identifying $s_{i}$ with the transposition $(i, i+1)$ for all $i \in[n]$, and extending multiplicatively. This is not the unique isomorphism and, as usual, we abuse notation by referring to the Coxeter system $(W, S)$ simply by $\mathfrak{S}(n)$. In the sequel, we write both $s_{i}$ and $i$ for the transposition $(i, i+1)$.

The Coxeter system $(\mathfrak{S}(n), S)$ has rank $n-1$ and its Coxeter graph is


Many of the concepts we have given in general Coxeter group theory can be reformulated in a simpler form for the symmetric group $\mathfrak{S}(n)$. In particular, we will need the following useful characterization of the Bruhat order (see, e.g., [47], Chap.1, for a proof). For $\sigma \in \mathfrak{S}(n)$, and $i \in[n]$, we let

$$
\left\{\sigma^{i, 1}, \ldots, \sigma^{i, i}\right\}_{<}:=\{\sigma(1), \ldots, \sigma(i)\}
$$

Theorem 0.4.1 Let $\sigma, \tau \in \mathfrak{S}(n)$. Then $\sigma \leq \tau$ if and only if $\sigma^{i, j} \leq \tau^{i, j}$ for all $1 \leq j \leq i \leq n-1$.

As in [49], we call an irreducible Coxeter system linear if it has Coxeter graph with no branch points, that is if it is isomorphic, for a certain $n$, to a Coxeter system ( $W, S=\left\{s_{1}, \ldots, s_{n}\right\}$ ) with:

$$
\begin{cases}m\left(s_{i}, s_{j}\right) \geq 3, & \text { if }|i-j|=1, \\ m\left(s_{i}, s_{j}\right)=2, & \text { if } 1<|i-j|<n-1 .\end{cases}
$$

(strictly linear if also $m\left(s_{1}, s_{n}\right)=2$, non-strictly otherwise). These are the Coxeter graphs associated respectively to a strictly and to a non-strictly linear Coxeter system:

where there is no restriction on the labels $m_{i, j}:=m\left(s_{i}, s_{j}\right)$. This class not only includes the symmetric groups, but also many of the classical Coxeter groups
such as those of type $B, F, H, \tilde{C}, I(m)$ (which are strictly linear) and those of type $\tilde{A}$ (which are non-strictly linear). See [39] for a complete description of classical Coxeter groups.

### 0.5 Kazhdan-Lusztig theory

In this section we introduce the basic elements of Kazhdan-Lusztig theory. All definitions and results appearing here are due to Kazhdan and Lusztig and their proofs can be found in [40] or [39, Chapter 7].

Kazhdan-Lusztig polynomials were originally introduced in terms of the Hecke algebra ([40]). Let $(W, S)$ be any Coxeter system. The Hecke Algebra $\mathcal{H}$ of $W$ over the ring of Laurent polynomials $\mathbb{Z}\left[q^{\frac{1}{2}}, q^{-\frac{1}{2}}\right]$ is the free $\mathbb{Z}\left[q^{\frac{1}{2}}, q^{-\frac{1}{2}}\right]$ module

$$
\mathcal{H}:=\bigoplus_{w \in W} \mathbb{Z}\left[q^{\frac{1}{2}}, q^{-\frac{1}{2}}\right] T_{w}
$$

with basis $\left\{T_{w}: w \in W\right\}$ and multiplication defined by:

$$
T_{s} T_{w}:= \begin{cases}T_{s w}, & \text { if } s \notin D_{L}(v)  \tag{3}\\ (q-1) T_{w}+q T_{s w}, & \text { if } s \in D_{L}(v)\end{cases}
$$

for all $w \in W$ and $s \in S$. Every element $T_{w}$ of the canonical basis of $\mathcal{H}$ is invertible; as $l(w)$ increases, however, the expression of the inverse gets more and more complicated and this is the reason why the family $\left\{R_{u, v}(q)\right\}$ of $R$-polynomials was defined, essentially as its coordinates with respect to the canonical basis of $\mathcal{H}$. More precisely, we have the following result.

Proposition 0.5.1 There exists a unique family $\left\{R_{u, w}(q)\right\}_{u, w \in W} \subseteq \mathbb{Z}[q]$ of polynomials satisfying

$$
\left(T_{w^{-1}}\right)^{-1}=(-1)^{l(w)} q^{-l(w)} \sum_{u \leq w}(-1)^{l(u)} R_{u, w}(q) T_{u},
$$

for all $w \in W$.
The polynomials $R_{u, v}$ which have been defined by the previous proposition are called the $R$-polynomials of $W$. It is easy to see that $\operatorname{deg}\left(R_{u, v}\right)=l(u, v)$ if $u \leq v$, and that $R_{u, v}(q)=1$ if $u=v$, for all $u, v \in W$. It is customary to let $R_{u, v}(q):=0$ if $u \not \leq v$. We then have the following result that follows from (3) and Proposition 0.5.1 (see [39, §7.5]).

Theorem 0.5.2 Let $u, v \in W$ and $s \in D_{L}(v)$. Then

$$
R_{u, v}(q)= \begin{cases}R_{s u, s v}(q), & \text { if } s \in D_{L}(u),  \tag{4}\\ q R_{s u, s v}(q)+(q-1) R_{u, s v}, & \text { if } s \notin D_{L}(u) .\end{cases}
$$

Note that the preceding theorem can be used to inductively compute the $R$ polynomials since $l(v s)<l(v)$. There is also a right version of Theorem 0.5.2.

It is sometimes convenient to use a related family of polynomials with nonnegative integer coefficients, called the $\widetilde{R}$-polynomials. For $u, v \in W$ we let $\widetilde{R}_{u, v}(q)$ be the unique polynomial such that

$$
\begin{equation*}
R_{u, v}(q)=q^{\frac{l(u, v)}{2}} \widetilde{R}_{u, v}\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right) \tag{5}
\end{equation*}
$$

It is not difficult to verify that this condition determines a monic polynomial $\tilde{R}_{u, v}(q) \in \mathbb{N}[q]$ of degree $l(u, v)$, satisfying the following recurrence relation, which is a consequence of Theorem 0.5.2.

Corollary 0.5.3 Let $u, v \in W$. Then $\widetilde{R}_{u, v}(q)=0$ if $u \not \leq v$ and $\widetilde{R}_{u, v}(q)=1$ if $u=v$. If $u<v$ and $s \in D_{L}(v)$ then

$$
\widetilde{R}_{u, v}(q)=\widetilde{R}_{s u, s v}(q)+\chi(s u \triangleright u) q \widetilde{R}_{u, s v}(q)
$$

Now we introduce a fundamental involution on $\mathcal{H}$. Define $\iota\left(q^{\frac{1}{2}}\right)=q^{-\frac{1}{2}}$ and $\iota\left(T_{w}\right)=\left(T_{w^{-1}}\right)^{-1}$ and combine these assignments to obtain a ring automorphism $\iota: \mathcal{H} \rightarrow \mathcal{H}$, which is clearly an involution. Now we look for a special basis of $\mathcal{H}$, again indexed by $W$, consisting of elements fixed by $\iota$. One may easily check that the elements

$$
C_{s}^{\prime}:=q^{-\frac{1}{2}}\left(T_{s}+T_{e}\right)
$$

are fixed by $\iota$. These are the first elements of the basis we are looking for.
Theorem 0.5.4 There exists a unique basis $\mathcal{C}^{\prime}=\left\{C_{w}^{\prime}: w \in W\right\}$ of $\mathcal{H}$ such that:

1. $\iota\left(C_{w}^{\prime}\right)=C_{w}^{\prime}$;
2. $C_{w}^{\prime}=q^{-\frac{l(w)}{2}} \sum_{u \leq w} P_{u, w}(q) T_{u}$;
3. $P_{u, w} \in \mathbb{Z}[q]$ has degree at most $\frac{1}{2}(l(u, w)-1)$ if $u<w$, and $P_{w, w}=1$.

The elements of the basis $\mathcal{C}^{\prime}$ are currently called Kazhdan-Lusztig elements and are usually denoted this way following the notation of [40], where they were first introduced. The polynomials $\left\{P_{u, v}(q)\right\}_{u, v \in W} \subseteq \mathbb{Z}[q]$ (where, for notational convenience, it is usual to set $P_{u, v}(q):=0$ if $\left.u \nless v\right)$ are the well known Kazhdan-Lusztig polynomials, or $P$-polynomials. As the coefficient of $q^{\frac{1}{2}(l(u, v)-1)}$ in $P_{u, v}(q)$ plays a very important role, we denote it, as customary, by $\mu(u, v)$ and we write $u \prec v$ if $\mu(u, v) \neq 0$.
The proof of the existence of the Kazhdan-Lusztig elements can be obtained by showing the recursive property they satisfy. This recurrence leads to the following multiplication.

Proposition 0.5.5 Let $s \in S$. Then

$$
C_{s} C_{w}= \begin{cases}\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right) C_{w}, & \text { if } s \in D_{L}(w) \\ C_{s w}+\sum_{s \in D_{L}(z)} \mu(z, w) C_{z}, & \text { if } s \notin D_{L}(w)\end{cases}
$$

for all $w \in W$.
Hence, given $w \in W$, we have

$$
C_{w}=C_{s} C_{s w}-\sum_{z: s \in D_{L}(z)} \mu(z, s w) C_{z} .
$$

for all $s \in D_{L}(w)$.

Both $R$-polynomials (and hence $\widetilde{R}$-polynomials) and Kazhdan-Lusztig polynomials could be equivalently introduced in a purely combinatorial way through the following results.

Theorem 0.5.6 Let $(W, S)$ be a Coxeter system. Then there is a unique family of polynomials $\left\{R_{u, v}(q)\right\}_{u, v \in W} \subseteq \mathbb{Z}[q]$ satisfying the following conditions:

1. $R_{u, v}(q)=0$ if $u \not \leq v$;
2. $R_{u, u}(q)=1$;
3. if $s \in D_{L}(v)$ then

$$
R_{u, v}(q)= \begin{cases}R_{s u, s v}(q), & \text { if } s \in D_{L}(u), \\ q R_{s u, s v}(q)+(q-1) R_{u, s v}(q), & \text { if } s \notin D_{L}(u)\end{cases}
$$

Theorem 0.5.7 Let $(W, S)$ be a Coxeter system. Then there is a unique family of polynomials $\left\{\widetilde{R}_{u, v}(q)\right\}_{u, v \in W} \subseteq \mathbb{Z}[q]$ satisfying the following conditions:

1. $\widetilde{R}_{u, v}(q)=0$ if $u \not \leq v$;
2. $\widetilde{R}_{u, u}(q)=1$;
3. if $s \in D_{L}(v)$ then

$$
\widetilde{R}_{u, v}(q)=\widetilde{R}_{s u, s v}(q)+\chi(s u \triangleright u) q \widetilde{R}_{u, s v}(q) .
$$

Theorem 0.5.8 Let $(W, S)$ be a Coxeter system. Then there is a unique family of polynomials $\left\{P_{u, v}(q)\right\}_{u, v \in W} \subseteq \mathbb{Z}[q]$ satisfying the following conditions:

1. $P_{u, v}(q)=0$ if $u \not \leq v$;
2. $P_{u, u}(q)=1$;
3. $\operatorname{deg}\left(P_{u, v}(q)\right) \leq \frac{1}{2}(l(u, v)-1)$, if $u<v$;
4. if $u \leq v$, then

$$
q^{l(u, v)} P_{u, v}\left(\frac{1}{q}\right)=\sum_{u \leq z \leq v} R_{u, z}(q) P_{z, v}(q) .
$$

The recursive relation for computing the Kazhdan-Lusztig polynomials is given in the following results.

Theorem 0.5.9 Let $(W, S)$ be a Coxeter system, $u, v \in W, u \leq v$, and $s \in$ $D_{L}(v)$. Then

$$
P_{u, v}(q)=q^{1-c} P_{s u, s v}(q)+q^{c} P_{u, s v}(q)-\sum_{z: s \in D_{L}(z)} q^{\frac{l(z, v)}{2}} \mu(z, s v) P_{u, z}(q)
$$

where $c=\chi(s u<u)$.
Corollary 0.5.10 Let $(W, S)$ be a Coxeter system, $u, v \in W, u<v$, and $s \in$ $D_{L}(v)$. Then $P_{u, v}(q)=P_{s u, v}(q)$.

Proposition 0.5 .5 and Theorems $0.5 .6,0.5 .7,0.5 .9$ and 0.5 .10 can also be reformulated in right versions.

In order to find a method for the computation of the dimensions of the intersection cohomology modules corresponding to Schubert varieties in $G / P$, where $P$ is a parabolic subgroup of the Kac-Moody group $G$, Deodhar ([27]) defined two parabolic analogues of Kazhdan-Lusztig and $R$-polynomials, which correspond to the roots of the equation $x^{2}=q+(q-1) x$. These polynomials are related to their ordinary counterparts in several ways; in particular, the parabolic Kazhdan-Lusztig polynomials of type -1 are the ordinary ones in the way of Proposition 0.5.13. But they also have direct application in different context. For example, they have connections to the theories of tilting modules ([54]), quantized Schur algebras ([61]) and Lie algebras (in [46], Leclerc and Thibon show that the Littlewood-Richardson coefficients are values at 1 of certain parabolic Kazhdan-Lusztig polynomials of type $q$ ). Despite this, there are very few explicit formulae for them.

We refer to $[27, \S \S 2-3]$ for the proofs of the two following result.

Theorem 0.5.11 Let $(W, S)$ be a Coxeter system, and $J \subseteq S$. Then, for each $x \in\{-1, q\}$, there is a unique family of polynomials $\left\{R_{u, v}^{J, x}(q)\right\}_{u, v \in W^{J}} \subseteq \mathbb{Z}[q]$ such that, for all $u, v \in W^{J}$ :

1. $R_{u, v}^{J, x}(q)=0$ if $u \not \leq v$;
2. $R_{u, u}^{J, x}(q)=1$;
3. if $u<v$ and $s \in D_{R}(v)$, then

$$
R_{u, v}^{J, x}(q)= \begin{cases}R_{u s, v s}^{J, x}(q), & \text { if } s \in D_{R}(u), \\ (q-1) R_{u, v s}^{J, x}(q)+q R_{u s, v s}^{J, x}(q), & \text { if } s \notin D_{R}(u) \text { and us } \in W^{J}, \\ (q-1-x) R_{u, v s}^{J, x}(q), & \text { if } s \notin D_{R}(u) \text { and us } \notin W^{J} .\end{cases}
$$

Theorem 0.5.12 Let $(W, S)$ be a Coxeter system, and $J \subseteq S$. Then, for each $x \in\{-1, q\}$, there is a unique family of polynomials $\left\{P_{u, v}^{J, x}(q)\right\}_{u, v \in W^{J}} \subseteq \mathbb{Z}[q]$, such that, for all $u, v \in W^{J}$ :

1. $P_{u, v}^{J, x}(q)=0$ if $u \not 又 v$;
2. $P_{u, u}^{J, x}(q)=1$;
3. $\operatorname{deg}\left(P_{u, v}^{J, x}(q)\right) \leq \frac{1}{2}(l(u, v)-1)$, if $u<v$;
4. if $u \leq v$, then

$$
q^{l(u, v)} P_{u, v}^{J, x}\left(\frac{1}{q}\right)=\sum_{z \in[u, v]_{J}} R_{u, z}^{J, x}(q) P_{z, v}^{J, x}(q)
$$

The polynomials $R_{u, v}^{J, x}(q)$ and $P_{u, v}^{J, x}(q)$ of Theorems 0.5.11 and 0.5.12 are called the parabolic $R$-polynomials and parabolic Kazhdan-Lusztig polynomials of $W^{J}$ of type $x$. By definition, $R_{u, v}^{\emptyset,-1}(q)\left(=R_{u, v}^{\emptyset, q}(q)\right)$ and $P_{u, v}^{\emptyset,-1}(q)\left(=P_{u, v}^{\emptyset, q}(q)\right)$ are the ordinary $R$-polynomials and Kazhdan-Lusztig polynomials of $W$.

Parabolic Kazhdan-Lusztig and $R$-polynomials are related to their ordinary counterparts also in the following way (see [27, Propositions 2.12 and Remark 3.8] for a proof).

Proposition 0.5.13 Let $(W, S)$ be a Coxeter system, $J \subseteq S$, and $u, v \in W^{J}$.
Then we have

$$
R_{u, v}^{J, x}(q)=\sum_{w \in W_{J}}(-x)^{l(w)} R_{w u, v}(q)
$$

for all $x \in\{-1, q\}$, and

$$
P_{u, v}^{J, q}(q)=\sum_{w \in W_{J}}(-1)^{l(w)} P_{w u, v}(q)
$$

(in particular, $\mu(u, v)$ is also the coefficient of $q^{\frac{1}{2}(l(u, v)-1)}$ in $\left.P_{u, v}^{J, q}(q)\right)$. Moreover, if $W_{J}$ is finite then

$$
P_{u, v}^{J,-1}(q)=P_{w_{0}^{J} u, w_{0}^{J} v}(q) .
$$

The Kazhdan-Lusztig polynomials of type $q$ have the following recursive formula (see [27, Proposition 3.10]), that will be used in the sequel.

Theorem 0.5.14 Let $(W, S)$ be a Coxeter system, $J \subseteq S$, and $u, v \in W^{J}$, $u \leq v$. Then for each $s \in D_{R}(v)$ we have

$$
P_{u, v}^{J, q}(q)=\tilde{P}-\sum_{w \in[u, v s]_{J}: s \in D_{R}(w)} \mu(w, v s) q^{\frac{1}{2} l(w, v)} P_{u, w}^{J, q}(q)
$$

where

$$
\tilde{P}= \begin{cases}P_{u s, v s}^{J, q}(q)+q P_{u, v s}^{J, q}(q), & \text { if } u s<u, \\ q P_{u, v s}^{J, q}(q)+P_{u, v s}^{J, q}(q), & \text { if } u<u s \in W^{J}, \\ 0, & \text { if } u<u s \notin W^{J} .\end{cases}
$$

Remark. It is easy to prove by induction on $l(v)$ that if $u s \notin W^{J}$ then any $P_{u, w}^{J, q}(q)$ in the sum of Theorem 0.5 .14 is 0 , and consequently the parabolic Kazhdan-Lusztig polynomial $P_{u, v}^{J, q}(q)$ is 0 . Recall that, if $u \leq v$, the ordinary Kazhdan-Lusztig polynomial $P_{u, v}(q)$ is always non-zero.

Corollary 0.5.15 Let $(W, S)$ be a Coxeter system, $J \subseteq S$, and $u, v \in W^{J}$, $u \leq v$. Then, for each $s \in D_{R}(v)$, we have

$$
P_{u, v}^{J, q}(q)=P_{u s, v}^{J, q}(q) .
$$

In particular, if $s \in D_{R}(v) \backslash D_{R}(u)$, then $\mu(u, v)=0$.

We refer to $[9,39]$ and $[40,27]$ for more details concerning general Coxeter group theory, and ordinary and parabolic Kazhdan-Lusztig polynomials.

### 0.6 Combinatorial invariance conjecture

One of the most famous conjecture in Kazhdan-Lusztig theory is certainly Lusztig's conjecture on the combinatorial invariance of Kazhdan-Lusztig polynomials. This long standing conjecture states that the Kazhdan-Lusztig polynomial $P_{u, v}(q)$ depends only on the isomorphism type of the interval $[u, v]$ as a poset.

Conjecture 0.6.1 (Lusztig) Let $(W, S)$ and $\left(W^{\prime}, S^{\prime}\right)$ be two Coxeter systems, $u, v \in W$ and $u^{\prime}, v^{\prime} \in W$. Suppose that $\Phi:[u, v] \rightarrow\left[u^{\prime}, v^{\prime}\right]$ is an isomorphism of posets (under Bruhat order). Then

$$
P_{x, y}(q)=P_{\Phi(x), \Phi(y)}(q)
$$

for all $x, y \in[u, v]$.
As a direct consequence of Theorem 0.5.8 and of (5), Conjecture 0.6 .1 can be reformulated both in terms of $R$ and $\widetilde{R}$-polynomials.

Corollary 0.6.2 Let $(W, S)$ and $\left(W^{\prime}, S\right)$ be two Coxeter systems, $u, v \in W$ and $u^{\prime}, v^{\prime} \in W^{\prime}$, and let

$$
\Phi:[u, v] \longrightarrow\left[u^{\prime}, v^{\prime}\right]
$$

be an isomorphism of posets. Then the following are equivalent:
i) $P_{x, y}(q)=P_{\Phi(x), \Phi(y)}(q)$ for all $x, y \in[u, v]$;
ii) $R_{x, y}(q)=R_{\Phi(x), \Phi(y)}(q)$ for all $x, y \in[u, v]$;
iii) $\widetilde{R}_{x, y}(q)=\widetilde{R}_{\Phi(x), \Phi(y)}(q)$ for all $x, y \in[u, v]$.

For many years there have been very few partial results to support it. This conjecture was known to be true for $[u, v]$ lattice (see [11]) and for $[u, v]$ of rank $\leq 4$. Precisely,

$$
\widetilde{R}_{u, v}(q)=\left\{\begin{array}{cc}
q^{l(u, v)}, & \text { if }[u, v] \text { is a lattice }  \tag{6}\\
q^{3}+q, & \text { if }[u, v] \text { is a 2-crown } \\
q^{4}+\frac{B_{2}(u, v)}{2} q^{2}, & \text { if } l(u, v)=4
\end{array}\right.
$$

where $B_{2}(u, v)$ is the number of paths from $u$ to $v$ of length 2 in the Bruhat graph of $W$. Recently in [17] Brenti proved that Conjecture 0.6 .1 is true when $W$ and $W^{\prime}$ are symmetric groups, and $u$ and $u^{\prime}$ are the identities of $W$ and $W^{\prime}$ (see Corollary 0.7.7).

In Section 2.3 we prove that Lusztig's conjecture holds when the Coxeter groups $W$ and $W^{\prime}$ are linear Coxeter groups, and the elements $v$ and $v^{\prime}$ are Boolean elements. All Chapter 4 is devoted to what is probably the most general result on the combinatorial invariance. We prove that Lusztig's conjecture is true when $u$ and $u^{\prime}$ are the identities of $W$ and $W^{\prime}$ with no restrictions on the Coxeter groups $W$ and $W^{\prime}$. The proof of this result is based on the concept of special matching, to which is devoted the following section.

### 0.7 Special matchings

In this section we follow [17] to define the special matchings of a poset, which are fundamental in Part II. We also collect the results of [17] that will be needed in the sequel for future references. Special matchings had already been considered in the literature by du Cloux ([30]) under the equivalent concept of compression labelings.

Remind that a matching of a graph $G$ with vertex set $V$ and edge set $E$ is an involution $M: V \rightarrow V$ such that $\{M(v), v\} \in E$ for all $v \in V$. A matching of a graph may be visualized by coloring with the same color all edges of the form $\{M(v), v\}$.

Definition. Let $P$ be a partially ordered set. We say that a matching $M$ of the Hasse diagram of $P$ is a special matching of $P$ if

$$
u \triangleleft v \Longrightarrow M(u) \leq M(v)
$$

for all $u, v \in P$ such that $M(u) \neq v$.
For example, the dotted matching of the following poset is a special matching

while the dashed one is not. For convenience, in some figures we do not draw the line of the covering relation between $v$ and $M(v)$. Note that a special matching has certain rigidity properties. For example, if $u \triangleleft v$ and $M(u) \triangleright u$, then $M(v) \triangleright v$ and $M(u) \triangleleft M(v)$.

The following result is the analogue of the Lifting Lemma (Lemma 0.3.4).
Lemma 0.7.1 (Lifting Lemma for special matchings) Let $M$ be a special matching of a locally finite ranked poset $P$, and let $u, v \in P, u \leq v$. Then

1. if $M(v) \triangleleft v$ and $M(u) \triangleleft u$ then $M(u) \leq M(v)$;
2. if $M(v) \triangleright v$ and $M(u) \triangleright u$ then $M(u) \leq M(v)$;
3. if $M(v) \triangleleft v$ and $M(u) \triangleright u$ then $M(u) \leq v$ and $u \leq M(v)$.

Lemma 0.7 .1 is actually a generalization of the Lifting Lemma and will play an important role in the sequel.

Now restrict our attention to the case where $P$ is a lower Bruhat interval of the symmetric group, namely an interval of the form $[e, v]$, with $v \in \mathfrak{S}(n)$. In this case we simply refer to a special matching of $[e, v]$ as a special matching of $v$. Every right or left descent of $v$ leads to a special matching of $v$ (this is
actually true in any Coxeter group). In fact, let $s_{i} \in D_{R}(v)$ and define the matching $\rho$ of $[e, v]$ by $\rho(u):=u s_{i}$, for all $u \in[e, v]$. The classical Lifting Lemma (Lemma 0.3.4) in particular implies that $\rho$ satisfies the axioms of a special matching. Analogously, the matching $\lambda$ defined by $\lambda(u):=s_{i} u$ for all $u \in[e, v]$ is a special matching whenever $s_{i} \in D_{L}(v)$.
The following is a further result on the rigidity of special matchings of permutations. It states that a special matching of a permutation is completely determined by how it acts on the atoms.

Lemma 0.7.2 Let $v \in \mathfrak{S}(n)$ and $M, N$ be two special matchings of $v$ such that $M(u)=N(u)$ for all $u \leq v$ with $l(u) \leq 1$. Then

$$
M(u)=N(u)
$$

for all $u \in[e, v]$.
The next result we are going to show, is a complete characterization of the special matchings of $v \in \mathfrak{S}(n)$. For this we firstly need some notation. For all $i \in[n-1]$ we denote respectively by $\lambda_{i}, \rho_{i}: \mathfrak{S}(n) \rightarrow \mathfrak{S}(n)$ the multiplications on the left and on the right respectively by $s_{i}$. In other words, $\lambda_{i}(v):=s_{i} v$ and $\rho_{i}(v):=v s_{i}$ for all $v \in \mathfrak{S}(n)$. Now fix $i \in[n-1]$, and let $J=[i]$ and $K=[i, n-1]$. Then we set

- $l_{i}(u):=u_{J} s_{i}{ }^{J} u$,
- $r_{i}(u):=u_{K} s_{i}{ }^{K} u$,
where $u=u_{J}{ }^{J} u$ and $u=u_{K}{ }^{K} u$ are the decompositions of $u$ relative to the parabolic subgroups $\mathfrak{S}(n)_{J}$ and $\mathfrak{S}(n)_{K}$ (see Theorem 0.3.7). We also denote by $\lambda_{i}, \rho_{i}, l_{i}$ and $r_{i}$ any restriction of these applications to a proper subset of $\mathfrak{S}(n)$.

Theorem 0.7.3 Let $v \in \mathfrak{S}(n)$ and $M$ be a special matching of $v$ with $M(e)=$ $s_{i}$. Then $M$ is either $\lambda_{i}, \rho_{i}, l_{i}$ or $r_{i}$.

We say that a special matching $M$ is of type $\lambda$ if $M=\lambda_{i}$ for some $i \in[n-1]$ and we similarly define special matchings of type $\rho$, of type $l$ and of type $r$. Note that a special special matching may have more than one type. In fact, for example, the unique matching of the trivial interval $\left[e, s_{i}\right]$ has all the types. The proof of Theorem 0.7.3 tells us also that special matchings which are not of type $\lambda$ or $\rho$ are quite rare. More precisely, we have the following results, that we state here for future references.

Corollary 0.7.4 Let $v \in \mathfrak{S}(n)$.

1. If $l_{i}$ is a special matching of $v$ then

$$
s_{i+1} s_{i} s_{i-1} \not \leq v .
$$

2. If $r_{i}$ is a special matching of $v$ then

$$
s_{i-1} s_{i} s_{i+1} \not \leq v
$$

Corollary 0.7.5 Let $u, v \in \mathfrak{S}(n), u \leq v, J=[i]$ and $K=[i, n-1]$.

1. Let $l_{i}$ be a special matching of $v$ and let $u=u_{1} u_{2}$ with $u_{1} \in \mathfrak{S}(n)_{J}$ and $u_{2} \in \mathfrak{S}(n)_{K}$. Then we have either $u_{1}=u_{J}$ or $u_{1}=u_{J} s_{i}$. In particular, in both cases,

$$
l_{i}(u)=u_{1} s_{i} u_{2}
$$

2. Let $r_{i}$ be a special matching of $v$ and let $u=u_{1} u_{2}$ with $u_{1} \in \mathfrak{S}(n)_{K}$ and $u_{2} \in \mathfrak{S}(n)_{J}$. Then we have either $u_{1}=u_{K}$ or $u_{1}=u_{J K} s_{i}$. In particular, in both cases,

$$
l_{i}(u)=u_{1} s_{i} u_{2}
$$

Using the classification of Theorem 0.7.3, Brenti proves the following result, which is the main theorem of [17].

Theorem 0.7.6 Let $v \in \mathfrak{S}(n)$ and $M$ be a special matching of $v$. Then, for all $u \leq v$,

$$
R_{u, v}(q)= \begin{cases}R_{M(u), M(v)}(q), & \text { if } M(u) \triangleleft u ; \\ q R_{M(u), M(v)}(q)+(q-1) R_{u, M(v)}(q), & \text { otherwise }\end{cases}
$$

and, equivalently,

$$
\widetilde{R}_{u, w}(q)=\widetilde{R}_{M(u), M(w)}(q)+\chi(M(u) \triangleright u) q \widetilde{R}_{u, M(w)}(q)
$$

Since, by definition, the set of the special matchings of $v$ depends only on the isomorphism type of $[e, v]$ as a poset, Theorem 0.7.6 is a partial result towards Lusztig conjecture on the combinatorial invariance (Conjecture 0.6.1).

Corollary 0.7.7 Let $v \in \mathfrak{S}(n)$ and $v^{\prime} \in \mathfrak{S}(m)$ be such that $[e, v] \cong\left[e, v^{\prime}\right]$ as posets. Then

$$
\begin{aligned}
& P_{u, v}(q)=P_{\varphi(u), v^{\prime}}(q) \\
& R_{u, v}(q)=R_{\varphi(u), v^{\prime}}(q) \\
& \widetilde{R}_{u, v}(q)=\widetilde{R}_{\varphi(u), v^{\prime}}(q)
\end{aligned}
$$

for all $u \leq v$ and all poset isomorphism $\varphi:[e, v] \rightarrow\left[e, v^{\prime}\right]$.
In Chapter 4 we generalize Theorem 0.7.6 to any Coxeter group, and hence we can prove the analogue of Corollary 0.7.7 for any Coxeter group.

## Part I

## Explicit formulae for Boolean elements

## Chapter 1

## Boolean elements

In this chapter, we intyroduce the Boolean elements and we give the preliminary results that make easier the combinatorics of these elements.

### 1.1 Definition and preliminary results

Definition. Let $(W, S)$ be any Coxeter system and let $t$ be a reflection in $W$. As in [49], we call $t$ a Boolean reflection if it admits a Boolean expression, which is, by definition, a reduced expression $s_{1} \ldots s_{n-1} s_{n} s_{n-1} \ldots s_{1}$ with $s_{h} \in S$ for all $h \in[n]$ and $s_{i} \neq s_{j}$ if $i \neq j$. Call any element $x \in W$ a Boolean element if it is smaller than a Boolean reflection.

We need the following lemma.
Lemma 1.1.1 Given a Coxeter system $(W, S)$, let $s, t_{1}, \ldots, t_{n} \in S, s \neq t_{i}$ for all $i \in[n]$, and $l\left(t_{1} \ldots t_{n}\right)=n$. Furthermore let $t_{i_{1}} \ldots t_{i_{h}}$ be a reduced subword of $t_{1} \ldots t_{n}$ such that st $t_{i_{1}} \ldots t_{i_{h}} \leq t_{1} \ldots t_{n} s$. Then $s$ commutes with every $t_{i_{1}}, \ldots, t_{i_{h}}$.

Proof. Since $s \neq t_{i}$ for all $i \in[n], s t_{i_{1}} \ldots t_{i_{h}}$ and $t_{1} \ldots t_{n} s$ are reduced expressions. Then there exists a reduced subword $t_{j_{1}} \ldots t_{j_{h+1}}$ of $t_{1} \ldots t_{n} s$ such that

$$
t_{j_{1}} \ldots t_{j_{h+1}}=s t_{i_{1}} \ldots t_{i_{h}}
$$

First of all, $t_{j_{h+1}}=s$ because $s$ must appear in $t_{j_{1}} \ldots t_{j_{h+1}}$ which is a subword of $t_{1} \ldots t_{n} s$ and $s \neq t_{i}$ for all $i \in[n]$. By Tits' Word Theorem $s t_{i_{1}} \ldots t_{i_{h}}$ and $t_{j_{1}} \ldots t_{j_{h}} s$ are linked by a sequence of braid moves. The analysis of this sequence give us
the assertion.
Let us start from $s t_{i_{1}} \ldots t_{i_{h}}$. We do all the braid moves until we encounter a braid move that involves $s$. There must be such a move in the sequence because at the end $s$ will be in the rightmost place. So we reach an expression of the following type:

$$
s t_{i_{1}^{\prime}} \ldots t_{i_{h}^{\prime}}
$$

and the next braid move involves $s$ and (necessarily) $t_{i_{1}^{\prime}}$. Being $t_{i_{2}^{\prime}} \neq s$, it must be $\alpha_{s, t_{i_{1}^{\prime}}}=s t_{i_{1}^{\prime}}$, namely $s$ commutes with $t_{i_{1}^{\prime}}$. So we do that move and we obtain $t_{i_{1}^{\prime}} s t_{i_{2}^{\prime}} \ldots t_{i_{h}^{\prime}}$.
At the $m^{\text {th }}$ step we reach an expression of the following type:

$$
t_{i_{1} \ldots} t_{i_{m-1}} s t_{i_{m}} \ldots t_{i_{h}}
$$

and we have proved that $s$ commutes with every $t_{i_{1}}, \ldots, t_{i_{m-1}}$. As before, we do all the following braid moves of the sequence till we encounter a move that involves $s$. Again there must be such a move in the sequence because at the end $s$ will be in the rightmost place. So we reach an expression of the following type:

$$
t_{i_{1}^{\prime}} \ldots t_{i_{m-1}^{\prime}} s t_{i_{m}^{\prime}} \ldots t_{i_{h}^{\prime}}
$$

If the following braid move involves $s$ and $t_{i_{m-1}^{\prime}}$ we do it and return to the $(m-1)^{t h}$ step. If it involves $s$ and $t_{i_{m}^{\prime}}$, since $s \neq t_{i_{m+1}^{\prime}}$, it must be $\alpha_{s, t_{i_{m}^{\prime}}}=s t_{i_{m}^{\prime}}$, namely $s$ commutes with $t_{i_{m}^{\prime}}$. We do the move obtaining

$$
t_{i_{1}^{\prime}} \ldots t_{i_{m}^{\prime}} s t_{i_{m+1}^{\prime}} \ldots t_{i_{h}^{\prime}}
$$

and we pass at the $(m+1)^{t h}$ step, having proved that $s$ commutes also with $t_{i_{m}^{\prime}}$.
At the end of the sequence of braid moves we obtain $t_{j_{1}} \ldots t_{j_{h}} s$ and we prove that $s$ commutes with every $t_{j_{1}}, \ldots, t_{j_{h}}$, that is with every $t_{i_{1}}, \ldots, t_{i_{h}}$.

The following lemma essentially says what one gains in Tits' Word Theorem (Theorem 0.3.6) by adding the hypothesis that the element $u \in W$ is Boolean. A short braid move is, by definition, a braid move of the shortest type (namely $\alpha_{s, s^{\prime}}=s s^{\prime}$ ). Given any $s \in S$ and any word $\bar{v} \in S^{*}$ (where $S^{*}$ denotes the free monoid on the set $S$ ), we denote by $\bar{v}(s)$ the number of occurrences of the letter $s$ in the word $\bar{v}$.

Lemma 1.1.2 Given a Coxeter system ( $W, S$ ), let $u \in W$ be a Boolean element and let $\bar{u}$ be a reduced expression of $u$ which is subword of the Boolean expression $s_{1} \ldots s_{n} \ldots s_{1}$. Then:

1. any other reduced expression $\underline{u}$ of $u$ which is a subword of $s_{1} \ldots s_{n} \ldots s_{1}$ is linked to $\bar{u}$ by a sequence of short braid moves;
2. any expression $\underline{u}$ of $u$ (not necessarily reduced) which is a subword of $s_{1} \ldots s_{n} \ldots s_{1}$ is linked to $\bar{u}$ by a sequence of short braid and nil moves.

Proof. 1). Let $i$ be the minimum of the $j \in[n]$ such that the dispositions of the factors $s_{j}$ in $\bar{u}$ and $\underline{u}$ are different (i.e. for every $h<i, \bar{u}\left(s_{h}\right)=\underline{u}\left(s_{h}\right)$ and $s_{h}$ appears on the same side in $\bar{u}$ and in $\underline{u}$ if $\bar{u}\left(s_{h}\right)=\underline{u}\left(s_{h}\right)=1$ ). Obviously $\bar{u}\left(s_{i}\right)=0$ if and only if $\underline{u}\left(s_{i}\right)=0$.
It is not possible that $\bar{u}\left(s_{i}\right) \neq \underline{u}\left(s_{i}\right)$. In fact, suppose $\bar{u}\left(s_{i}\right)=2, \underline{u}\left(s_{i}\right)=1$; after cancelling from $\bar{u}$ and $\underline{u}$ the factors $s_{h}$ for $h<i$ and the factor $s_{i}$ in the same position, we would obtain two reduced expressions of the same element, one with and the other without factors $s_{i}$.
So $\underline{u}\left(s_{i}\right)=\bar{u}\left(s_{i}\right)=1$. After cancelling the factors $s_{h}$ for $h<i$ from $\bar{u}$ and $\underline{u}$, we obtain two reduced expressions of the same element, one with only one factor $s_{i}$ at the leftmost place and the other with only one factor $s_{i}$ at the rightmost place. Since $s_{i} \neq s_{j}$ for every $i \neq j$, by Lemma 1.1.1 $s_{i}$ commutes with every $s_{j}, j>i$, that occurs in $\bar{u}$. Iterating this procedure, we get the assertion.
2). Let $\underline{u}=t_{1} \ldots t_{q}\left(t_{i} \in S\right)$ and let $r$ be such that $t_{1} \ldots t_{r}$ is reduced, but $t_{1} \ldots t_{r} t_{r+1}$ is not. By the Exchange Property (Theorem 0.3.1), there exists a unique $i$ such that $t_{1} \ldots t_{r} t_{r+1}=t_{1} \ldots \widehat{t_{i}} \ldots t_{r}$ (obviously this last expression is reduced) and $t_{i+1} \ldots t_{r} t_{r+1}=t_{i} t_{i+1} \ldots t_{r}$. Since these are both reduced subwords of $s_{1} \ldots s_{n} \ldots s_{1}$, by 1) they are linked by a sequence of short braid moves. So from the expression $t_{1} \ldots t_{i} t_{i+1} \ldots t_{r} t_{r+1} \ldots t_{q}$, using only short braid moves, we can reach the expression $t_{1} \ldots t_{i} t_{i} t_{i+1} \ldots t_{r} t_{r+2} \ldots t_{q}$ and then we can do a nil move. By iterating this procedure, using only short braid and nil moves, we obtain a reduced expression of $u$ which is subword of $s_{1} \ldots s_{n} \ldots s_{1}$. Hence the assertion follows by 1).

Corollary 1.1.3 Given a Coxeter system $(W, S)$, let $\bar{u}$, $\underline{u}$ be two reduced expressions of the same Boolean element $u \in W$ which are both subwords of a Boolean expression $s_{1} \ldots s_{n} \ldots s_{1}$. Then $\bar{u}\left(s_{i}\right)=\underline{u}\left(s_{i}\right)$ for all $i \in[n]$.
Proof. It is straightforward from Lemma 1.1.2.

Now we state two technical results that are easy to prove. We assume that the linear Coxeter systems have Coxeter graphs of the types in Section 0.4.

Proposition 1.1.4 Let $(W, S)$ be a strictly linear Coxeter system and let $t \in W$ be a Boolean reflection. Then $t$ admits a Boolean expression of one of the following types:

$$
\begin{aligned}
& \text { 1. } s_{a} s_{a-1} \ldots s_{i+1} s_{b} s_{b+1} \ldots s_{i-1} s_{i} s_{i-1} \ldots s_{b+1} s_{b} s_{i+1} \ldots s_{a-1} s_{a} \text {, } \\
& \text { 2. } s_{b} s_{b+1} \ldots s_{i-1} \quad s_{a} s_{a-1} \ldots s_{i+1} s_{i} s_{i+1} \ldots s_{a-1} s_{a} s_{i-1} \ldots s_{b+1} s_{b} \text {, }
\end{aligned}
$$

for appropriate $0<b \leq i \leq a \leq n$.
Proposition 1.1.5 Let $\left(W, S=\left\{s_{1}, \ldots, s_{n}\right\}\right)$ be a non-strictly linear Coxeter system and let $t \in W$ be a Boolean reflection. Then, up to a "rotation" of the indices of the generators (that is up to adding a fixed $r \in[n-1]$ to their indices and taking the indices modulo $n$ ), $t$ admits a Boolean expression of one of the following types:

$$
\begin{aligned}
& \text { 1. } s_{a} s_{a-1} \ldots s_{i+1} \\
& s_{b} s_{b+1} \ldots s_{i-1} s_{i} s_{i-1} \ldots s_{b+1} s_{b}
\end{aligned} s_{i+1} \ldots s_{a-1} s_{a} \text {, },
$$

for appropriate $0<b \leq i \leq a \leq n$. If $s_{i} \leq t$ for all $i \in[n]$, we can assume $a \neq(i+1)$ in 1), $b \neq(i-1)$ in 2).

### 1.2 Notation on Boolean permutations

Let us specialize to the case $W=\mathfrak{S}(n+1)$. Recall that the set $S$ of Coxeter generators is the set of simple transpositions $\left\{s_{i}=(i, i+1)\right.$ for all $\left.i \in[n]\right\}$, the set of reflections is the set of transpositions

$$
T(\mathfrak{S}(n+1))=\{(i, j): 1 \leq i<j \leq n+1\}
$$

and the transposition $(i, j)$ admits $s_{i} s_{i+1} \cdots s_{j-2} s_{j-1} s_{j-2} \cdots s_{i+1} s_{i}$ as a reduced expression. So every reflection in the symmetric group is Boolean and an element $v$ is Boolean if and only if $v$ is smaller than the top transposition $(1, n+1)$. Equivalently, $v$ is Boolean if and only if it admits a reduced expression which is a subword of $s_{1} \cdots s_{n-1} s_{n} s_{n-1} \cdots s_{1}$. Note that a Boolean element can have several reduced expressions which are all subwords of $s_{1} \cdots s_{n-1} s_{n} s_{n-1} \cdots s_{1}$.

Now we introduce the notation that will be used in Chapter 2.
$n$-Boolean sequences. After Corollary 1.1.3, we denote by $u_{i}$ the number of occurrences of $s_{i}$ in any reduced expression of $u$ which is a subword of the Boolean expression $s_{1} \ldots s_{n} \ldots s_{1}$ of $(1, n+1)$. It is sometimes convenient to handle Boolean elements in terms of sequences. So we introduce a well-defined surjective map $\phi$ from the interval $[e,(1, n+1)]$ to the set of the $n$-Boolean sequences by sending $u$ to $\left(u_{1}, \ldots, u_{n}\right)$. An $n$-Boolean sequence is a sequence $\left(x_{1}, \ldots, x_{n}\right)$ of $n$ numbers chosen in $\{0,1,2\}$ that avoids the pattern $|2,0|$, where $|2,0|$-avoidance means that there does not exist an $i \in[n-1]$ such that $\left(x_{i}, x_{i+1}\right)=(2,0)$ and that $x_{n} \neq 2$. All properties are easily checked.
Given a $n$-Boolean sequence $x=\left(x_{1}, \ldots, x_{n}\right)$, we define:

$$
\begin{aligned}
l(x) & =\sum_{i \in[n]} x_{i} \\
p(x) & =\left|\left\{i \in[n-1]: x_{i}=1, x_{i+1} \neq 0\right\}\right|
\end{aligned}
$$

Then the cardinality of the preimage of the sequence $x$ is equal to $2^{p(x)}$ and $l(u)=l(\phi(u))$ for all $u \in[e,(1, n+1)]$.
If we endow the range with the component-wise partial order, then it is easy to check that $\phi$ is a morphism of posets.

Now we introduced the notation that will be used in Chapter 3.
The maps $\phi_{R}(u, v)$ and $\phi_{L}(u, v)$. For convenience, for all $J \subseteq S$, we identify $J$ with the set $\left\{i \in[n]: s_{i} \in J\right\}$. Let $w$ be a Boolean permutation of $\mathfrak{S}(n+1)$. The permutation $w$ can have several reduced expressions which are subwords of $s_{1} \cdots s_{n-1} s_{n} s_{n-1} \cdots s_{1}$. We consider all these expressions as obtained from $s_{1} \cdots s_{n-1} s_{n} s_{n-1} \cdots s_{1}$ by deleting some letters. For example, consider the Boolean permutation $w \in \mathfrak{S}(4)$ equal to $(1,2)(3,4)$ in the cyclic notation. Then $w$ has the following two reduced expressions which are obtained from $s_{1} s_{2} s_{3} s_{2} s_{1}$ in two different ways:
(1) $s_{3} s_{1}=\widehat{s_{1}} \widehat{s_{2}} s_{3} \widehat{s_{2}} s_{1}$
(2) $s_{1} s_{3}=s_{1} \widehat{s_{2}} s_{3} \widehat{s_{2}} \widehat{s_{1}}$
where $\widehat{s}$ means that $s$ has been deleted. We say that $s_{1}$ is "on the right" in (1) and "on the left" in (2).

Given two Boolean permutations $u, v \in \mathfrak{S}(n+1)^{J}, u \leq v$, we want to construct two $(2 \times n)$-rectangular tableaux with entries in $\left\{0,1_{l}, 1_{r}, 2\right\}$.

Suppose first that $v \not \leq s_{1} \cdots s_{n-1} s_{n}$. After Lemma 1.1.2, we choose

- the unique reduced expression $\bar{v}$ of the permutation $v$ which is a subword of $s_{1} \cdots s_{n-1} s_{n} s_{n-1} \cdots s_{1}$ and satisfies the condition that, for all $k \in[n-1]$ such that $\bar{v}\left(s_{k}\right)=1$ and $\bar{v}\left(s_{k+1}\right)=0$, the letter $s_{k}$ is on the right;
- the unique Boolean expression $\bar{u}$ of $u$ which is a subword of $\bar{v}$ and satisfies the further condition that, for all $k \in[n-1]$ such that $\bar{u}\left(s_{k}\right)=1, \bar{u}\left(s_{k+1}\right)=$ 0 and $\bar{v}\left(s_{k}\right)=2$, the letter $s_{k}$ is on the right.

We call $(\bar{u}, \bar{v})$ the right Boolean expressions of $(u, v)$. Then $\phi_{R}(u, v)$ is the $2 \times n$-rectangular tableau

where $v_{i}$ (respectively $u_{i}$ ) is $2,1_{l}, 1_{r}$, or 0 according as to whether $\bar{v}$ (respectively $\bar{u})$ has two letters $s_{i}$, one letter $s_{i}$ on the left, one letter $s_{i}$ on the right or no letters $s_{i}$. Finally, we mark the $i$-th column with $\circ$ if $i \in J$, with $\times$ if $i \notin J$. The dual conditions give rise to the left Boolean expressions of $(u, v)$ and to the $(2 \times n)$-rectangular tableau $\phi_{L}(u, v)$.
For convenience, in both tableaux $\phi_{R}(u, v)$ and $\phi_{L}(u, v)$, we set $v_{n}=1_{l}$ if $\bar{v}\left(s_{n}\right)=1$ and $u_{n}=1_{l}$ if $\bar{u}\left(s_{n}\right)=1$.
For example, if $v=s_{1} s_{3} s_{5} s_{6} s_{7} s_{8} s_{6} s_{5} s_{4}$ and $u=s_{7} s_{5} s_{3}$ are permutations of $\mathfrak{S}(9)$, then the right Boolean expressions $(\bar{u}, \bar{v})$ of $(u, v)$ are

$$
\begin{aligned}
& -\bar{v}=s_{3} s_{5} s_{6} s_{7} s_{8} s_{6} s_{5} s_{4} s_{1} ; \\
& -\bar{u}=s_{3} s_{7} s_{5} ;
\end{aligned}
$$

the left Boolean expressions $(\bar{u}, \bar{v})$ of $(u, v)$ are

$$
\begin{aligned}
-\bar{v} & =s_{1} s_{3} s_{5} s_{6} s_{7} s_{8} s_{6} s_{5} s_{4} ; \\
-\bar{u} & =s_{3} s_{5} s_{7} ;
\end{aligned}
$$

and, assuming $J=\{2,4,6\}$, we have

$$
\begin{aligned}
& \times 0 \times 0 \times 0 \times \times
\end{aligned}
$$

If $v \leq s_{1} \cdots s_{n-1} s_{n}$, we define the right and the left Boolean expressions to be equal, with all the letters on the left. Thus, in this case, $\phi_{R}(u, v)=\phi_{L}(u, v)$ and all non-zero entries are equal to $1_{l}$.

Furthermore, we introduce the following notation. Choose one of the two tableaux $\phi_{R}(u, v), \phi_{L}(u, v)$. We denote by

the cardinality of the set:

$$
\left\{i \in[n]: \begin{array}{l}
\left(v_{i}, v_{i+1}, v_{i+2}, v_{i+3}, \ldots\right)=(a, b, c, d, \ldots), \\
\left(u_{i}, u_{i+1}, u_{i+2}, u_{i+3}, \ldots\right)=(\alpha, \beta, \gamma, \delta, \ldots)
\end{array}\right\} .
$$

We let $a, b, c, d, \ldots, \alpha, \beta, \gamma, \delta, \ldots \in\left\{0,1_{l}, 1_{r}, 2, \emptyset, \nsim 2, *\right\}$ where by $\emptyset$ (respectively $\nsupseteq$ ) we mean that the entry must be $\neq 0$ (respectively $\neq 2$ ) and where $*$ stands for any entry. As above, if necessary, we use $\circ$ or $\times$ to further require that a column belong to $J$ or not. In the previous example,

$$
\begin{aligned}
& \stackrel{\times}{\stackrel{+}{\mid 1_{l} *}} \\
& 11_{l} 0=2
\end{aligned}
$$

both in $\phi_{R}(u, v)$ and $\phi_{L}(u, v)$. In other words, we are counting the sub-tableaux of $\phi_{R}(u, v)$ or of $\phi_{L}(u, v)$ matching $1_{l} 0$.

Now, let $v$ be a Boolean permutation in $\mathfrak{S}(n+1)$ and let $\bar{v}$ be any of its reduced expressions which are subwords of $s_{1} \cdots s_{n-1} s_{n} s_{n-1} \cdots s_{1}$. By Propo-

| $v_{i-1}$ | $v_{i}$ | $v_{i+1}$ |
| :---: | :---: | :---: |
| $*$ | 0 | $*$ |
| $1_{l}$ | $1_{l}$ | $*$ |
| $2,1_{l}$ | 2 | $*$ |
| $*$ | $1_{r}$ | $\neq 0$ |
| $1_{l}$ | $1_{r}$ | 0 |

Table 1.1:
sition 0.3.2 and Tits' Word Theorem (Theorem 0.3.6), we have that $s_{i} \notin D_{L}(v)$
if and only if we are in one of the (mutually exclusive) possibilities in Table 1.2, where $v_{i-1}, v_{i}, v_{i+1}$ encode the types of occurrences of $s_{i-1}, s_{i}, s_{i+1}$ in $\bar{v}$, and where $*$ stands for any entry. In particular, if $v$ is a Boolean permutation in $\mathfrak{S}(n+1)^{J}$, then this must be true for all $i \in J$.

## Chapter 2

## $R$-polynomials and

## Kazhdan-Lusztig polynomials

In this Chapter we give some closed explicit product formulae valid in the case that the indexing elements are Boolean. In particular, for any Coxeter system, we compute the $R$-polynomials, and for any linear Coxeter system we compute the Kazhdan-Lusztig polynomials, the Kazhdan-Lusztig elements and the intersection homology Poincaré polynomials. Moreover the formula for the KazhdanLusztig polynomials allows us to prove Lusztig's conjecture of the combinatorial invariance foe Boolean elements and to list all pairs $(u, v)$ of Boolean elements with $u \prec v$, namely with $\mu(u, v) \neq 0$.
Throughout this chapter, when the Coxeter group $W$ is the symmetric group, we make use of the notion of $n$-Boolean sequence we introduced in Section 1.2.

## $2.1 \quad R$-polynomials

Recall that for any $s \in S$ and any word $\bar{x} \in S^{*}$ (where $S^{*}$ denotes the free monoid on the set $S$ ), we denote by $\bar{x}(s)$ the number of occurrences of the letter $s$ in the word $\bar{x}$.

Theorem 2.1.1 Given any Coxeter system $(W, S)$, let $u, v \in W$ be Boolean elements, $u \leq v$. Fix a reduced expression $\bar{v}$ of $v$ which is a subword of a Boolean expression $s_{1} \ldots s_{n} \ldots s_{1}$ and a reduced expression $\bar{u}$ of $u$ which is a
subword of $\bar{v}$. Then

$$
R_{u, v}(q)=(q-1)^{l(u, v)-2 a}\left(q^{2}-q+1\right)^{a}
$$

where

$$
\left.a=\left\lvert\,\left\{i \in[n]: \begin{array}{l}
\bar{v}\left(s_{i}\right)=2 \\
\bar{u}\left(s_{i}\right)=0
\end{array} \text { and } m\left(s_{i}, s_{j}\right)=2, \forall j>i \text { such that } \bar{u}\left(s_{j}\right) \neq 0\right\}\right. \right\rvert\, .
$$

In particular, if $W=\mathfrak{S}(n+1)$ and $s_{1}, \ldots, s_{n}$ are the usual Coxeter generators of $\mathfrak{S}(n+1)$, this means that:

$$
a=\left|\left\{i \in[n]: \begin{array}{l}
v_{i}=2 \\
u_{i}=0
\end{array} \quad u_{i+1}=0 \quad\right\}\right| .
$$

Proof. We proceed by induction on $n$, the result being clear for $n=1$. If $\bar{v}\left(s_{1}\right)=\bar{u}\left(s_{1}\right)=0$, we conclude right away by induction since $u \leq v \leq$ $s_{2} \ldots s_{n-1} s_{n} s_{n-1} \ldots s_{2}$. So we suppose $\bar{v}\left(s_{1}\right) \neq 0$ and focus our attention on the number and the position of the occurrences of $s_{1}$ in $\bar{v}$ and $\bar{u}$. We have to consider the following cases, in which $\hat{s_{1}}$ means that $s_{1}$ has been deleted and in which we do not bother about $s_{i}, i \neq 1$.

Then by Theorem 0.5.2 we get $R_{u, v}(q)=R_{s_{1} u, s_{1} v}(q)$ and we conclude by induction since $s_{1} u \leq s_{1} v \leq s_{2} \ldots s_{n-1} s_{n} s_{n-1} \ldots s_{2}$.

Then by Theorem 0.5.2 we get $R_{u, v}(q)=q R_{s_{1} u, s_{1} v}+(q-1) R_{u, s_{1} v}(q)$ and we conclude by induction since $s_{1} u \not \leq s_{1} v$ and $u \leq s_{1} v \leq s_{2} \ldots s_{n-1} s_{n} s_{n-1} \ldots s_{2}$.

Like $\mathrm{a}_{1}$ ) using the right version of Theorem 0.5.2.
$\left.\mathrm{b}_{2}\right)\left\{\begin{array}{l}\bar{v}=\hat{s_{1}} \ldots \hat{\ldots} . \hat{s_{n}} s_{n} \ldots \hat{\ldots} . \hat{s_{1}} \\ \bar{u}=\hat{s_{1}} \ldots \hat{s_{1}} \hat{\ldots} . s_{n} \ldots \hat{\ldots} . \hat{s_{1}}\end{array}\right.$
Like $\mathrm{a}_{2}$ ) using the right version of Theorem 0.5.2.
$\left.\mathrm{c}_{1}\right)\left\{\begin{array}{l}\bar{v}=s_{1} \ldots \hat{\ldots} \hat{\ldots} . . s_{n} \ldots \hat{\ldots} \hat{\ldots} . s_{1} \\ \bar{u}=s_{1} \ldots \hat{\ldots} \ldots s_{n} \ldots \hat{\ldots} . . s_{1}\end{array}\right.$
$R_{u, v}(q)=R_{s_{1} u, s_{1} v}(q)=R_{s_{1} u s_{1}, s_{1} v s_{1}}(q)$ and we conclude by induction since $s_{1} u s_{1} \leq s_{1} v s_{1} \leq s_{2} \ldots s_{n-1} s_{n} s_{n-1} \ldots s_{2}$.
$\left.\mathrm{c}_{2}\right)\left\{\begin{array}{l}\bar{v}=s_{1} \ldots \hat{\ldots} \hat{\ldots} . . s_{n} \ldots \hat{\ldots} \hat{\ldots} . s_{1} \\ \bar{u}=s_{1} \ldots \hat{\ldots} \ldots . . s_{n} \ldots \hat{\ldots} . . . \hat{s}_{1}\end{array}\right.$
$R_{u, v}(q)=R_{s_{1} u, s_{1} v}(q)=q R_{s_{1} u s_{1}, s_{1} v s_{1}}(q)+(q-1) R_{s_{1} u, s_{1} v s_{1}}$ and we conclude by induction since $s_{1} u s_{1} \not \leq s_{1} v s_{1}, s_{1} u \leq s_{1} v s_{1} \leq s_{2} \ldots s_{n-1} s_{n} s_{n-1} \ldots s_{2}$.
$\left.\mathrm{c}_{3}\right)\left\{\begin{array}{l}\bar{v}=s_{1} \ldots \hat{\ldots} \ldots . . s_{n} \ldots \hat{\ldots} \hat{\ldots} . s_{1} \\ \bar{u}=\hat{s_{1}} \ldots \hat{\ldots} \ldots . s_{n} \ldots \hat{\ldots} . . s_{1}\end{array}\right.$
Like $c_{2}$.

We have to distinguish two subcases:

1) $s_{1} u \nless s_{1} v$

Then we get
$R_{u, v}(q)=q R_{s_{1} u, s_{1} v}(q)+(q-1) R_{u, s_{1} v}=(q-1)\left[q R_{u s_{1}, s_{1} v s_{1}}(q)+(q-1) R_{u, s_{1} v s_{1}}\right]$
and we conclude by induction since $u s_{1} \not \subset s_{1} v s_{1}, u \leq s_{1} v s_{1} \leq s_{2} \ldots s_{n-1} s_{n} s_{n-1} \ldots s_{2}$.
2) $s_{1} u \leq s_{1} v$

Then we get

$$
\begin{gathered}
R_{u, v}(q)=q R_{s_{1} u, s_{1} v}(q)+(q-1) R_{u, s_{1} v}= \\
=q R_{s_{1} u s_{1}, s_{1} v s_{1}}(q)+(q-1)\left[q R_{u s_{1}, s_{1} v s_{1}}(q)+(q-1) R_{u, s_{1} v s_{1}}(q)\right]= \\
=\left(q^{2}-q+1\right) R_{u, s_{1} v s_{1}}(q)
\end{gathered}
$$

being, by Lemma 1.1.1, $u=s_{1} u s_{1}$ and $u s_{1} \not \subset s_{1} v s_{1}$. So we conclude by induction since $u \leq s_{1} v s_{1} \leq s_{2} \ldots s_{n-1} s_{n} s_{n-1} \ldots s_{2}$.

Call $u^{\prime}$ and $v^{\prime}$ the elements which are represented by the expressions we obtain from $\bar{u}$ and $\bar{v}$ by deleting all the letters $s_{1}$. In every case, except in subcase 2) of case $c_{4}$ ), we have

$$
R_{u, v}(q)=(q-1)^{\bar{v}\left(s_{1}\right)-\bar{v}\left(s_{1}\right)} R_{u^{\prime}, v^{\prime}}(q)
$$

Recall that we are in subcase 2) of case $\mathrm{c}_{4}$ ) when $\bar{v}\left(s_{1}\right)=2, \bar{u}\left(s_{1}\right)=0$ and $s_{1} u \leq s_{1} v$, namely, by Lemma 1.1.1, when $\bar{v}\left(s_{1}\right)=2, \bar{u}\left(s_{1}\right)=0$ and $s_{1}$ commutes with every $s_{j} j>1$ such that $\bar{u}\left(s_{j}\right) \neq 0$. In this case

$$
R_{u, v}(q)=\left(q^{2}-q+1\right) R_{u^{\prime}, v^{\prime}}(q)
$$

The result follows by iterating this procedure.
Example 1 Let us calculate the $R$-polynomial indexed by $u=s_{1} s_{2} s_{5} s_{1}$ and $v=s_{1} s_{2} s_{3} s_{4} s_{5} s_{6} s_{4} s_{3} s_{2} s_{1}$ in $\mathfrak{S}(7)$. We immediately find that $l(u, v)=6$ and $a=|\{3\}|$, and therefore

$$
R_{u, v}(q)=(q-1)^{4}\left(q^{2}-q+1\right)
$$

As a corollary of Theorem 2.1.1, we give the proof of Conjecture 7.7 of [15].
Corollary 2.1.2 Let $u, v \in \mathfrak{S}(n)$ be such that $u \leq v \leq(i, j)$ for some $i, j \in[n]$, $i<j$. Then there exists $a \in \mathbb{N}$ such that

$$
R_{u, v}(q)=(q-1)^{a}\left(q^{2}-q+1\right)^{\frac{1}{2}[l(u, v)-a]}
$$

Proof. It is straightforward from Theorem 2.1.1. In fact the transposition $(i, j)$ is a Boolean reflection of Boolean expression $s_{i} s_{i+1} \ldots s_{j-2} s_{j-1} s_{j-2} \ldots s_{i+1} s_{i}$ (where, as always, $s_{k}=(k, k+1)$ for all $k$ ).

We think that it is worthwhile to mention the following equivalence that deals with the $R$-polynomials which are product of factors of types $(q-1)$ and $\left(q^{2}-q+1\right)$, such as those of Theorem 2.1.1.

Theorem 2.1.3 Given a Coxeter system $(W, S)$, let $w \in W$. Then the following are equivalent:

1. $a(u, s v)=a(s u, s v)+1$ for all $u, v \leq w$ and $s \in S$ such that $u<s u \leq$ $s v<v$;
2. $R_{u, v}(q)=(q-1)^{a(u, v)}\left(q^{2}-q+1\right)^{\frac{1}{2}[l(u, v)-a(u, v)]}$ for all $u \leq v \leq w$;
where, for $x, y \in W, x \leq y,(q-1)^{a(x, y)}$ is the largest power of $(q-1)$ that divides $R_{x, y}(q)$.
Proof. Let us prove that 1) implies 2) by induction on $l(v)$. Let $s \in D_{L}(v)$. If $s \in D_{L}(u)$ or $s \notin D_{L}(u)$ but $s u \nless s v$ then we conclude by induction. Otherwise
$R_{u, v}(q)=q R_{s u, s v}(q)+(q-1) R_{u, s v}(q)$ that, by inductive assumption, is equal to $q\left[(q-1)^{a(s u, s v)}\left(q^{2}-q+1\right)^{\frac{1}{2}[l(s u, s v)-a(s u, s v)]}\right]+(q-1)\left[(q-1)^{a(u, s v)}\left(q^{2}-q+\right.\right.$ $\left.1)^{\frac{1}{2}[l(u, s v)-a(u, s v)]}\right]$. By hypothesis, this polynomial is equal to $(q-1)^{a(s u, s v)}\left(q^{2}-\right.$ $q+1)^{\frac{1}{2}[l(s u, s v)-a(s u, s v)]}\left[q+(q-1)^{2}\right]$.
Conversely fix (if there are) $s \in S$ such that $u<s u \leq s v<v$. Then $R_{u, v}(q)=$ $q R_{s u, s v}(q)+(q-1) R_{u, s v}(q)=q\left[(q-1)^{a(s u, s v)}\left(q^{2}-q+1\right)^{\frac{1}{2}[l(s u, s v)-a(s u, s v)]}+(q-\right.$ 1) $\left[(q-1)^{a(u, s v)}\left(q^{2}-q+1\right)^{\frac{1}{2}[l(u, s v)-a(u, s v)]}\right]$. But $R_{u, v}(q)=(q-1)^{a(u, v)}\left(q^{2}-q+\right.$ $1)^{\frac{1}{2}[l(u, v)-a(u, v)]}$ and an easy argument of divisibility shows that this is possible only if $a(u, s v)=a(s u, s v)+1$.

### 2.2 Kazhdan-Lusztig polynomials

Theorem 2.2.1 Let $u$ and $v$ be Boolean elements in $\mathfrak{S}(n+1), u \leq v$. Then

$$
P_{u, v}(q)=(1+q)^{b}
$$

where

$$
b=\left|\left\{k \in[n]: \begin{array}{ll}
v_{k}=2 & v_{k+1}=2 \\
& u_{k+1}=0
\end{array}\right\}\right|
$$

Proof. Fix a reduced expression $\bar{v}$ of $v$ which is a subword of the Boolean expression $s_{1} \ldots s_{n} \ldots s_{1}$ of $(1, n+1)$ and a reduced expression $\bar{u}$ of $u$ which is a subword of $\bar{v}$. Let us focus our attention on the number and the position of the factors $s_{1}$ in $\bar{v}$ and $\bar{u}$. We consider the following cases:
a) $v_{1}=u_{1}=1$.

We may assume that the letter $s_{1}$ is at the leftmost place in $\bar{v}$ and $\bar{u}$. Then, by Theorem 0.5.9, we get $P_{u, v}(q)=P_{s_{1} u, s_{1} v}(q)+q P_{u, s_{1} v}(q)=P_{s_{1} u, s_{1} v}(q)$ since $u \not \leq s_{1} v$.
b) $v_{1}=1, u_{1}=0$.

We may assume that the letter $s_{1}$ is at the leftmost place in $\bar{v}$. Then, by Corollary 0.5.10, we get $P_{u, v}(q)=P_{s_{1} u, v}(q)$ and we conclude that $P_{u, v}(q)=P_{u, s_{1} v}(q)$ as in a).
c) $v_{1}=u_{1}=2$.
$P_{u, v}(q)=P_{s_{1} u, s_{1} v}(q)+q P_{u, s_{1} v}(q)=P_{s_{1} u, s_{1} v}(q)$ since $u \not \leq s_{1} v$. So, as in a), we get $P_{u, v}(q)=P_{s_{1} u s_{1}, s_{1} v s_{1}}(q)$.
d) $v_{1}=2, u_{1}=1$.

We may assume that the letter $s_{1}$ is at the leftmost place in $\bar{u}$. By Corollary 0.5.10, $P_{u, v}(q)=P_{s_{1} u, v}(q)$ and we are in case e).
e) $v_{1}=2, u_{1}=0$.

We must distinguish two subcases:

1) $s_{1} u \neq u s_{1}$

By Lemma 1.1.1, this happens if and only if $u_{2} \neq 0$, or, equivalently, if and only if $s_{1} \bar{u} s_{1}$ is reduced. By Corollary 0.5.10 (first left and then right version), we get $P_{u, v}(q)=P_{s_{1} u, v}(q)=P_{s_{1} u s_{1}, v}(q)$ and, as in c), we get $P_{u, v}(q)=P_{u, s_{1} v s_{1}}(q)$.
2) $s_{1} u=u s_{1}$

Concerning the factors $s_{2}$, we have $u_{2}=0$ and two possibilities for $v$ :
i) $v_{2}=1$,
ii) $v_{2}=2$,
(necessarily $v_{2} \neq 0$ since $v_{1}=2$ ).
In i), we may assume that the letter $s_{2}$ is at the leftmost place in $\bar{v}$. Then $s_{2} \in D_{L}(v)$. So $P_{u, v}(q)=P_{s_{2} u, v}(q)$ and we are in case e) 1). We get $P_{u, v}(q)=P_{s_{2} u, s_{1} v s_{1}}(q)$. As to the factors $s_{2}$, we are in case a) and we get $P_{u, v}(q)=P_{u, s_{2} s_{1} v s_{1}}(q)$ finding that also the factors $s_{2}$ give no contribution. In ii), we get

$$
P_{u, v}(q)=q P_{s_{1} u, s_{1} v}(q)+P_{u, s_{1} v}(q)-\sum_{z: s_{1} \in D_{L}(z)} q^{\frac{l(z, v)}{2}} \mu\left(z, s_{1} v\right) P_{u, z}(q)
$$

By the fact that $s_{1}$ commutes with every $s_{i}$ that occurs in $\bar{u}$ and by Corollary 0.5.10, we get $P_{s_{1} u, s_{1} v}=P_{u s_{1}, s_{1} v}=P_{u, s_{1} v}$ and as in b) we get $P_{u, s_{1} v}=P_{u, s_{1} v s_{1}}$. So

$$
P_{u, v}(q)=(1+q) P_{u, s_{1} v s_{1}}(q)-\sum_{z: s_{1} \in D_{L}(z)} q^{\frac{l(z, v)}{2}} \mu\left(z, s_{1} v\right) P_{u, z}(q) .
$$

Now we claim that $\left\{z: u \leq z<s_{1} v, s_{1} \in D_{L}(z)\right\} \subseteq\left\{z: s_{2} \not \leq z\right\}$. In fact, $z<s_{1} v$ implies that $z$ admits a reduced expression $\bar{z}^{\prime} s_{1}$ with $\bar{z}^{\prime}\left(s_{1}\right)=0$. Since $s_{1} \in D_{L}(z), s_{1} \bar{z}^{\prime} s_{1}$ is not reduced and so, by the Exchange Property,
we get that $s_{1} \bar{z}^{\prime} s_{1}$ and $\bar{z}^{\prime}$ represent the same element, as $s_{1} \bar{z}^{\prime}$ is reduced. Applying Lemma 1.1.1 to $s_{1} \bar{z}^{\prime}=\bar{z}^{\prime} s_{1}$, we obtain that $s_{1}$ commutes with every letter that occurs in $\bar{z}^{\prime}$, namely $\bar{z}^{\prime}\left(s_{2}\right)=0$.
Therefore $s_{2} \in D_{L}\left(s_{1} v\right) \backslash D_{L}(z)$, and we find that

$$
\operatorname{deg} P_{z, s_{1} v}=\operatorname{deg} P_{s_{2} z, s_{1} v} \leq \frac{1}{2}\left(l\left(z, s_{1} v\right)-2\right)
$$

(since $\left.s_{2} z \neq s_{1} v\right)$. So $\mu\left(z, s_{1} v\right)=0$ for all $z$ in the sum and this gives $P_{u, v}(q)=(1+q) P_{u, s_{1} v s_{1}}(q)$.

In all cases, the $P$-polynomial indexed by $u$ and $v$ is equal to the $P$-polynomial indexed by the elements that we obtain from $\bar{u}$ and $\bar{v}$ by erasing all the factors $s_{1}$, except in cases d) and e) when they fall under the case e)-2)-ii). In these cases we get a factor $(1+q)$.
By iterating this procedure, the result follows.

We illustrate Theorem 2.2 .1 with an example.
Example 2 Let $W=\mathfrak{S}(8), u=s_{1} s_{5} s_{7}$ and $v=s_{1} s_{2} s_{3} s_{4} s_{5} s_{6} s_{7} s_{6} s_{5} s_{3} s_{2} s_{1}$. Then

$$
\begin{aligned}
& \phi(v)=(2,2,2,1,2,2,1) \\
& \phi(u)=(1,0,0,0,1,0,1)
\end{aligned}
$$

There are exactly 3 sub-tableaux of the type

| 2 | 2 |
| :--- | :--- |
|  | 0 |

in

\section*{| 2 | 2 | 2 | 1 | 2 | 2 |
| :--- | :--- | :--- | :--- | :--- | :--- |}

Therefore $P_{u, v}(q)=(1+q)^{3}$.
Note that, similarly, by Theorem 2.1.1, the number of sub-tableaux of the type

computes the $R$-polynomial $R_{u, v}(q)$.

Now we extend this result to other Coxeter systems. The same argument of the proof of Theorem 2.2.1 holds for every Coxeter system till we encounter the
case e)-2), where we strongly use the special properties of the symmetric group. So we need to proceed in a different way.
We show how Theorems 2.1.1 and 2.2.1, in conjunction with Lemma 1.1.2, imply the result for strictly linear Coxeter systems. First we need the following lemma, where we use the same symbols $s_{1}, \ldots, s_{m}$ for both the generators of $W$ and the generators of $\mathfrak{S}(m+1)$.

Lemma 2.2.2 Let $\left(W, S=\left\{s_{1}, \ldots, s_{m}\right\}\right)$ be a strictly linear Coxeter system. Let $t \in W$ be a Boolean reflection with Boolean expression $\bar{t}$. Consider the map $\psi:[e, t]_{W} \longrightarrow \mathfrak{S}(m+1)$ defined as follows: if $z \in[e, t]_{W}$ admits the reduced expression $\bar{z}$ which is a subword of $\bar{t}$, then $\psi(z)$ is the element of $\mathfrak{S}(m+1)$ represented by the same expression $\bar{z}$. Then $\psi$ is an isomorphism of posets from $[e, t]_{W}$ to $[e, \psi(t)]_{\mathfrak{S}(m+1)}$.

Proof. The map $\psi$ is well defined: in fact, by Lemma 1.1.2, any two such reduced expression of the same $z \in W$ are linked by short braid moves, and $W$ and $\mathfrak{S}(m+1)$ share the same short braid moves. Moreover, the expression $\bar{t}=t_{1} \ldots t_{n-1} t_{n} t_{n-1} \ldots t_{1}$ is reduced also in $\mathfrak{S}(m+1)$. In fact, suppose, by contradiction, that there exists $k \in[n]$ such that $t_{1} \ldots t_{n-1} t_{n} t_{n-1} \ldots t_{k-1}$ is reduced while $t_{1} \ldots t_{n-1} t_{n} t_{n-1} \ldots t_{k}$ is not. Then, clearly, $t_{k} \ldots t_{n-1} t_{n} t_{n-1} \ldots t_{k}$ is not reduced (by hypothesis, $t_{i} \neq t_{j}$ if $i \neq j$ ). Hence, by Lemma 1.1.1, $t_{k}$ commutes with $t_{j}$ for all $j>k$ in $\mathfrak{S}(m+1)$, and so also in $W$, and this is a contradiction because $\bar{t}$ is reduced in $W$. This means that $\bar{t}$ is a Boolean expression of the Boolean reflection $\psi(t)$ of $\mathfrak{S}(m+1)$. Now Lemma 1.1.2 implies that $l(z)=l(\psi(z))$, for all $z \in[e, t]_{W}$, and that $\psi$ is an isomorphism of posets from $[e, t]_{W}$ to $[e, \psi(t)]_{\mathfrak{S}(m+1)}$ by the characterization of the Bruhat order in terms of reduced expressions.

Theorem 2.2.3 Let $\left(W, S=\left\{s_{1}, \ldots, s_{m}\right\}\right)$ be a strictly linear Coxeter system. Let $u, v \in W$ be such that $u \leq v \leq t$, where $t$ is a Boolean reflection. Then

$$
P_{u, v}(q)=P_{\psi(u), \psi(v)}(q),
$$

where $\psi$ is as in Lemma 2.2.2, and $P_{\psi(u), \psi(v)}(q)$ can be computed as in Theorem 2.2.1.

Proof. First of all we fix a Boolean expression $\bar{t}$ of $t$, a reduced expression $\bar{v}$ of $v$ which is a subword of $\bar{t}$ and a reduced expression $\bar{u}$ of $u$ which is a subword of $\bar{v}$.

Recall that, if an element $z$ has a reduced expression $\bar{z}$ which is a subword of $\bar{v}$, then the map $\psi$ sends $z$ to the element of $\mathfrak{S}(m+1)$ represented by the same expression $\bar{z}$. Theorem 2.1 .1 shows that the $R$-polynomial depends only on the chosen reduced expression and on the commutation relations between the generators of the Coxeter system. So for all $x, y \in[u, v]_{W}, R_{x, y}(q)=R_{\psi(x), \psi(y)}(q)$. Finally property 4) of Theorem 0.5.8, in conjunction with Lemma 2.2.2, implies that the same equality also holds for the $P$-polynomials.

Example 3 Let $\left(W, S=\left\{s_{1}, s_{2}, s_{3}, s_{4}\right\}\right)$ be a strictly linear Coxeter system, $v=s_{4} s_{1} s_{2} s_{3} s_{2} s_{1} s_{4}, u=s_{4} s_{1}$. Then $\psi(v)=s_{4} s_{1} s_{2} s_{3} s_{2} s_{1} s_{4}=s_{1} s_{2} s_{3} s_{4} s_{3} s_{2} s_{1} \in$ $\mathfrak{S}(5), \psi(u)=s_{4} s_{1} \in \mathfrak{S}(5)$, and $P_{u, v}(q)=P_{\psi(u), \psi(v)}(q)=(1+q)^{2}$.

The following result deals with the non-strictly linear Coxeter systems.
Theorem 2.2.4 Let $\left(W, S=\left\{s_{1}, \ldots, s_{m}\right\}\right)$ be a non-strictly linear Coxeter system. Let $u, v \in W$ be such that $u \leq v \leq t$ where $t$ is a Boolean reflection that we can assume such that $s_{i} \leq t$ for all $i \in[m]$. Then there exists $b \in \mathbb{N}$ such that:

$$
P_{u, v}(q)=(1+q)^{b} .
$$

Fix a Boolean expression $\bar{t}=t_{1} \ldots t_{n-1} t_{n} t_{n-1} \ldots t_{1}$ for $t$ of the type shown in Proposition 1.1.5, a reduced expression $\bar{v}$ of $v$ which is a subword of $\bar{t}$ and a reduced expression $\bar{u}$ of $u$ which is a subword of $\bar{v}$. Suppose that $t_{j}$ is, together with $t_{2}$, the only other generator that does not commute with $t_{1}$. Then $P_{u, v}(q)=$ $(1+q)^{b^{\prime}} P_{u^{\prime}, v^{\prime}}(q)$, where $u^{\prime}$ and $v^{\prime}$ are the elements represented by the expressions we obtain by erasing all the letters $t_{1}$ in $\bar{u}$ and $\bar{v}$, and where

$$
b^{\prime}=\left\{\begin{array}{lc}
1, & \text { if } \bar{v}\left(t_{1}\right)=2, \bar{u}\left(t_{2}\right)=0 \text { and } \bar{u}\left(t_{j}\right)=0 \\
0, & \text { otherwise }
\end{array}\right.
$$

Then compute $P_{u^{\prime}, v^{\prime}}(q)$ as in Theorem 2.2 .3 (there are no longer occurrences of $\left.t_{1}\right)$.

Proof. We can repeat the same argument of the proof of Theorem 2.2.1, replacing $s_{1}$ with $t_{1}$, till we encounter the case e)-2), that now means that $\bar{v}$ has a letter $t_{1}$ both at the rightmost and at the leftmost place while $\bar{u}$ has no letters $t_{1}, t_{2}, t_{j}$. So we get:

$$
P_{u, v}(q)=q P_{t_{1} u, t_{1} v}(q)+P_{u, t_{1} v}(q)-\sum_{z: t_{1} \in D_{L}(z)} q^{\frac{l(z, v)}{2}} \mu\left(z, t_{1} v\right) P_{u, z}(q)
$$

By the fact that $t_{1}$ commutes with every $t_{i}$ that occurs in $\bar{u}$ and by Corollary 0.5.10, we get $P_{t_{1} u, t_{1} v}=P_{u t_{1}, t_{1} v}=P_{u, t_{1} v}$ and as in b) we get $P_{u, t_{1} v}=$ $P_{u, t_{1} v t_{1}}$. So

$$
P_{u, v}(q)=(1+q) P_{u, t_{1} v t_{1}}(q)-\sum_{z: t_{1} \in D_{L}(z)} q^{\frac{l(z, v)}{2}} \mu\left(z, t_{1} v\right) P_{u, z}(q) .
$$

Now we claim that $\left\{z: u \leq z<t_{1} v, t_{1} \in D_{L}(z)\right\} \subseteq\left\{z: t_{2} \not \leq z, t_{j} \not \leq z\right\}$. In fact, $z<t_{1} v$ implies that $z$ admits a reduced expression $\bar{z}^{\prime} t_{1}$ with $\bar{z}^{\prime}\left(t_{1}\right)=0$. Since $t_{1} \in D_{L}(z), t_{1} \bar{z}^{\prime} t_{1}$ is not reduced and so, by the Exchange Property, we get that $t_{1} \bar{z}^{\prime} t_{1}$ and $\bar{z}^{\prime}$ represent the same element, as $t_{1} \bar{z}^{\prime}$ is reduced. Applying Lemma 1.1.1 to $t_{1} \bar{z}^{\prime}=\bar{z}^{\prime} t_{1}$, we obtain that $t_{1}$ commutes with every letter that occurs in $\bar{z}^{\prime}$, namely $\bar{z}^{\prime}\left(t_{2}\right)=\bar{z}^{\prime}\left(t_{j}\right)=0$.
Therefore $t_{2} \in D_{L}\left(t_{1} v\right) \backslash D_{L}(z)$, and we find that

$$
\operatorname{deg} P_{z, t_{1} v}=\operatorname{deg} P_{t_{2} z, t_{1} v} \leq \frac{1}{2}\left(l\left(z, t_{1} v\right)-2\right)
$$

(since $\left.t_{2} z \neq t_{1} v\right)$. So $\mu\left(z, t_{1} v\right)=0$ for all $z$ in the sum and this gives $P_{u, v}(q)=$ $(1+q) P_{u, t_{1} v t_{1}}(q)$.
Now, since $u^{\prime} \leq v^{\prime} \leq t_{2} \ldots t_{n-1} t_{n} t_{n-1} \ldots t_{2}$, we can think of our elements as in the strictly linear Coxeter system ( $W^{\prime}, S \backslash\left\{t_{1}\right\}$ ).

Remarks. It is worthwhile to remark the following facts.

1. If the Coxeter system is not irreducible, and $S=\bigcup S_{i}$ is the decomposition into irreducible components, then the expression $t_{1} \ldots t_{n-1} t_{n} t_{n-1} \ldots t_{1}$ is reduced only if all the generators $t_{j}$ belong to the same $S_{i}$.
2. If $W=\mathfrak{S}(n)$, it is easy to see that a Boolean permutation $v$ is always covexillary (3412 avoiding). Therefore, the polynomial $P_{u, v}(q)$ can also be computed using the algorithm given in [44]. However, it seems to be difficult to derive the explicit formulae of Theorem 2.2.1 from this algorithm if $v<(1, n)$.
3. The results in this section do not hold for general Coxeter systems. In fact, let $(W, S)$ be a Coxeter system such that $S$ contains $s_{1}, s_{2}, s_{3}$ and $r$ with $m\left(s_{i}, s_{j}\right)=2$ for all $i \neq j, m\left(s_{i}, r\right) \geq 3$ for all $i$. Then $P_{u, v}(q)=1+2 q$, where $v=s_{1} s_{2} r s_{3} r s_{2} s_{1}, u=s_{3} s_{2} s_{1}$.

### 2.3 Combinatorial invariance

In this section we prove that Lusztig's conjecture of the combinatorial invariance (Conjecture 0.6.1) is true for Boolean elements in strictly linear Coxeter groups. More precisely, we prove that, given two Boolean elements $u$ and $v$ in a strictly linear Coxeter group $W$, the polynomial $P_{u, v}(q)$ can be easily computed from $l(u, v), c_{1}(u, v)$ and $c_{2}(u, v)$, where

$$
\begin{aligned}
c_{i}(u, v) & :=\left|C_{i}(u, v)\right| \\
C_{i}(u, v) & :=\{z \in[u, v]: l(z, v)=i\},
\end{aligned}
$$

for $i=1,2$. The elements of $C_{1}(u, v)$ are the coatoms of $[u, v]$. Furthermore, let $g_{i}(u, v)=\left|G_{i}(u, v)\right|$ and $h_{i}(u, v)=\left|H_{i}(u, v)\right|$, where

$$
\begin{aligned}
G_{i}(u, v) & :=\left\{z \in[u, v]: z^{-1} v \in T(W), l(z, v)=(1+2 i)\right\}, \\
H_{i}(u, v) & :=\left\{z \in[u, v]: u^{-1} z \in T(W), l(u, z)=(1+2 i)\right\},
\end{aligned}
$$

for all possible $i \in \mathbb{N}$. Thanks to the following theorem due to Dyer [32], they are all combinatorial invariants of $[u, v]$ as a poset.

Theorem 2.3.1 Let $(W, S)$ be a Coxeter system, $u, v \in W$. The isomorphism type of the poset $[u, v]$ determines the isomorphism type of its Bruhat graph.

As we can deduce from (6), if $l(u, v) \leq 4$, the $R$-polynomial $R_{u, v}(q)$ depends only on $g_{i}(u, v)$ and $h_{i}(u, v)$. At the end of this section we show that this is not true in general, and we give a counterexample. The smallest $\mathfrak{S}(n)$ in which we can find a counterexample for Boolean elements is $\mathfrak{S}(10)$.

Let us first consider the case $u$ and $v$ Boolean elements in $\mathfrak{S}(n+1)$. To simplify notation, we set

$$
X_{l, m}^{j, k}:=\left|\left\{i \in[n] ; \begin{array}{ll}
v_{i}=j & v_{i+1}=k \\
u_{i}=l & u_{i+1}=m
\end{array}\right\}\right| .
$$

In particular, $X_{1, \neq 0}^{2, *}$ means that $v_{i}=2, v_{i+1}$ can be any number, $u_{i}=1$ and $u_{i+1}$ must be different from 0 . We write, respectively, $a(u, v)$ and $b(u, v)$ for the exponents in Theorem 2.1.1 and in Theorem 2.2.1, and we always omit the dependence on $(u, v)$ when no confusion arises.
In the proof of the following results, we use Tits' Word Theorem (Theorem 0.3.6)
and Lemma 1.1.2 without explicit mention.
Proposition 2.3.2 Let $u$ and $v$ be Boolean elements in $\mathfrak{S}(n+1)$, $u \leq v$. Then

$$
\begin{align*}
& c_{1}=l+b-a  \tag{2.1}\\
& c_{2}=\frac{l}{2}(l-1)+\frac{b-a}{2}(b-a+2 l-3)-b \tag{2.2}
\end{align*}
$$

Proof. Equation (2.1) follows from a result of Brenti [11], valid in any Coxeter system $W$ and for any $x, y \in W$. It states that $c_{1}(x, y)$ is equal to the coefficient of $q$ in $P_{x, y}(q)$ (in this case $b$ by Theorem 2.2.1) minus $(-1)^{l}$ times the coefficient of $q$ in $R_{x, y}(q)$ (in this case $(-1)^{l+1}(l-a)$ by Theorem 2.1.1).
Fix a reduced expression $\bar{v}$ of $v$ which is a subword of the Boolean expression $s_{1} \ldots s_{n} \ldots s_{1}$ of $(1, n+1)$ and a reduced expression $\bar{u}$ of $u$ which is a subword of $\bar{v}$. Then $\bar{u}$ is obtained from $\bar{v}$ by deleting $l$ letters. We have that

$$
c_{2}=|A|-\frac{|B|}{2}+|C|
$$

where:

- $A$ is the set of the reduced expressions $\bar{z}$ we can obtain from $\bar{v}$ by deleting only 2 letters of those we deleted to obtain $\bar{u}$;
- $B \subset A \times A$ is the set of pairs $(\bar{z}, \underline{z})$ of distinct expressions in $A$ such that $\bar{z}$ and $\underline{z}$ are linked by short braid moves, and so represent the same element;
- $C$ is the set of the reduced expressions $\bar{z}$ such that:
$-\bar{z}$ is obtained from $\bar{v}$ by deleting 2 letters, $s_{i}$ and $s_{j}$, such that at least one of them, say $s_{i}$, is not deleted in $\bar{u}$;
$-\bar{z}$ does not represent an element already represented by an expression in $A$;
- $\bar{u}$ is linked by short braid moves to a subword of $\bar{z}$.

Let us calculate $|A|,|B|$ and $|C|$.
$A)$. Let $\bar{z}$ be an expression we obtain from $\bar{v}$ by deleting two factors, say $s_{i}$ and $s_{j}$, of those we deleted to obtain $\bar{u}$. It fails to be reduced if and only if for at least one between $i$ and $k$, say $i$, we have $\left(z\left(s_{i-1}\right), z\left(s_{i}\right)\right)=(2,0)$. If $i=j$, this happens only if $\left(v_{i-1}, v_{i}\right)=(2,2)$ and $u_{i}=0$. If $i \neq j$, this happens only if $\left(v_{i-1}, v_{i}\right)=(2,1)$ and $u_{i}=0$; in this case, the other factor $s_{j}$ we are deleting
can be any of the other letters of $\bar{v}$ that are deleted in $\bar{u}$, except $s_{i-1}$. These are $l-2$ if $u_{i-1}=1, l-3$ if $u_{i-1}=0$. Being careful not to count twice the case $\left(z\left(s_{i-1}\right), z\left(s_{i}\right), z\left(s_{k-1}\right), z\left(s_{k}\right)\right)=(2,0,2,0)$, we get:

$$
|A|=\binom{l}{2}-\left(X_{*, 0}^{2,2}+\sum_{k=3}^{2+X_{0,0}^{2,1}}(l-k)+\sum_{k=2+X_{0,0}^{2,1}}^{1+X_{1,0}^{2,1}+X_{0,0}^{2,1}}(l-k)\right)
$$

that, being $X_{*, 0}^{2,2}=b$ by Theorem 2.2.1, becomes:

$$
|A|=\binom{l}{2}-\left(b+\sum_{k=3}^{1+X_{*, 0}^{2,1}}(l-k)\right)-\left(l-2-X_{0,0}^{2,1}\right) .
$$

$B)$. Let $\bar{z}$ and $\underline{z}$ be two different expression in $A$ linked by braid moves. Necessarily, to obtain $\bar{z}$ and $\underline{z}$, we have deleted letters of the same type, say $s_{i}$ and $s_{j}$. Suppose that we have deleted the $s_{i}$ on the left to obtain $\bar{z}$ and on the right to obtain $\underline{z}$ (so necessarily $v_{i}=2$ ). If $\bar{z}$ and $\underline{z}$ are linked by braid moves, then $z_{i+1}=0$. But $v_{i+1} \neq 0$ because $v_{i}=2$, and so $j$ must be $i+1$. Hence $\frac{|B|}{2}=X_{0,0}^{2,1}$.
$C$ ). Necessarily $v_{i}=2, u_{i}=1$ and $u_{i+1}=0$, while $z_{i+1} \neq 0$ otherwise $\bar{z}$ would represent an element already represented by an expression in $A$. The element $c$ of expression $\bar{c}$ equal to $\bar{v}$ with only the $s_{j}$ deleted is a coatom. In fact it is reduced, otherwise it should be $v_{j}=1$ and $j=i+1$ ( $\bar{z}$ is reduced), but $z_{i+1} \neq 0$. Conversely, we obtain an element of those we are now counting from every coatom $c$ with $c_{i}=2$ deleting the letter $s_{i}$ not deleted in $\bar{u}$. The number of such coatoms is $\left(c_{1}-2\right)$ for all $i$ such that $v_{i}=2, u_{i}=1$ and $u_{i+1}=0$. Being careful to count without repetition, we get:

$$
|C|=\sum_{k=2}^{X_{1,0}^{2, *}+1}\left(c_{1}-k\right)
$$

that, by (2.1), becomes:

$$
|C|=\sum_{k=2}^{X_{1,0}^{2, *}+1}(l+b-a-k)
$$

Being $X_{0,0}^{2, *}=a$ by Theorem2.1.1, our assertion is proved.

Now we are able to prove the main theorem of this section.
Theorem 2.3.3 Let $(W, S)$ be a strictly linear Coxeter system, $u$ and $v$ be Boolean elements of $W$. Then $R_{u, v}(q)=(q-1)^{l-2 a}\left(q^{2}-q+1\right)^{a}$ and $P_{u, v}(q)=$ $(1+q)^{b}$ where

$$
\begin{aligned}
a & =2 l+\frac{c_{1}}{2}\left(c_{1}-5\right)-c_{2}, \\
b & =l+\frac{c_{1}}{2}\left(c_{1}-3\right)-c_{2} .
\end{aligned}
$$

Proof. If $W=\mathfrak{S}(n+1)$, the result follows combining (2.1) and (2.2). Otherwise, by the proof of Theorem 2.2.3, $[u, v]$ is poset-isomorphic to a certain interval in $\mathfrak{S}(n)$, for an appropriate $n$, and share the same Kazhdan-Lusztig polynomial with it. This proves our assertion.

Finally we show that considering only the $g_{i}$ and the $h_{i}$ is not the right way to tackle Lusztig's conjecture. In fact, we have the following example.

Example 2.3.4 Let $W=\mathfrak{S}(10)$,

$$
\begin{array}{ll}
v=s_{1} s_{2} s_{3} s_{4} s_{5} s_{6} s_{7} s_{8} s_{9} s_{4} s_{3} s_{2} s_{1}, & v^{\prime}=s_{1} s_{2} s_{3} s_{4} s_{5} s_{6} s_{7} s_{8} s_{9} s_{8} s_{7} s_{5} s_{4} s_{2} s_{1} \\
u=s_{1} s_{4}, & u^{\prime}=s_{1} s_{4} s_{7} s_{9}
\end{array}
$$

Then

$$
\begin{array}{ll}
\phi(v)=(2,2,2,2,1,1,1,1,1) & \phi\left(v^{\prime}\right)=(2,2,1,2,2,1,2,2,1) \\
\phi(u)=(1,0,0,1,0,0,0,0,0) & \phi\left(u^{\prime}\right)=(1,0,0,1,0,0,1,0,1)
\end{array}
$$

and $l=11, c_{1}=g_{0}=12, h_{0}=10, h_{1}=4$, and $g_{1}=g_{i}=h_{i}=0$, for $i>1$, for both the intervals $[u, v]$ and $\left[u^{\prime}, v^{\prime}\right]$. However $P_{u, v}=(1+q)^{2}$ while $P_{u^{\prime}, v^{\prime}}=(1+q)^{3}$. Of course, this agrees with the result in Theorem 2.3.3 since $c_{2}(u, v)=63$ while $c_{2}\left(u^{\prime}, v^{\prime}\right)=62$.

### 2.4 The top coefficient

In this section we classify all those Kazhdan-Lusztig polynomials indexed by Boolean elements in a linear Coxeter system ( $W, S$ ) which have the highest possible degree. These particular polynomials play a fundamental role in the
construction of the Kazhdan-Lusztig representations (see [40]). Moreover they appear in the recursive property of Theorem 0.5.9, and so Corollaries 2.4.1, 2.4.2 and 2.4.3 have applications in the computation of generic Kazhdan-Lusztig polynomials (see [23, 24]).
Let us treat first the case $W=\mathfrak{S}(n+1)$, and let us handle the Boolean elements in $\mathfrak{S}(n+1)$ in terms of $n$-Boolean sequences (see Section 1.2).

Corollary 2.4.1 Let $u, v \in \mathfrak{S}(n+1)$ be Boolean elements such that $l(u, v)>1$.
Then $u \prec v$ if and only if there exist $1 \leq l_{1}<l_{2}<n$ such that

$$
\begin{array}{ll}
v_{k}=u_{k}, & \text { if } 1 \leq k<l_{1}, \\
v_{k}=2 \text { and } u_{k}=1, & \text { if } k=l_{1}, \\
v_{k}=2 \text { and } u_{k}=0, & \text { if } l_{1}<k \leq l_{2}, \\
v_{k}=u_{k}, &
\end{array}
$$

Proof. The proof comes from the analysis of the proof of Theorem 2.2.1.
Fix a reduced expression $\bar{v}$ of $v$ which is a subword of $s_{1} \ldots s_{n} \ldots s_{1}$ and a reduced expression $\bar{u}$ of $u$ which is a subword of $\bar{v}$. To simplify, we define $P_{j}$ to be the Kazhdan-Lusztig polynomial indexed by the elements having as reduced expressions $\bar{u}$ and $\bar{v}$ with all the letters $s_{1}, \ldots, s_{j}$ deleted. For example, if $\bar{v}=s_{1} s_{2} s_{3} s_{4} s_{3} s_{1}$ and $\bar{u}=s_{1} s_{4}$, then $P_{2}=P_{s_{4}, s_{3} s_{4} s_{3}}(q)$.
Suppose that $v_{k}=u_{k}$, for $1 \leq k<l_{1}$, and $v_{l_{1}}>u_{l_{1}}$. Then $P_{u, v}(q)=P_{l_{1}-1}$ and $P_{l_{1}-1}$ is a Kazhdan-Lusztig polynomial indexed by elements whose difference of the length is $l(u, v)$. If $\left(v_{l_{1}}, v_{l_{1}+1}, u_{l_{1}}, u_{l_{1}+1}\right) \notin\{(2,2,0,0),(2,2,1,0)\}$, then $P_{u, v}(q)=P_{l_{1}}$ but $P_{l_{1}}$ is indexed by elements whose difference of the length is $<l(u, v)$, and so $P_{u, v}(q)$ cannot have maximum degree allowed (by hypothesis $l(u, v)>1$ and so $P_{l_{1}}$ is not indexed by equal elements if $\left.v_{l_{1}}=u_{l_{1}}+1\right)$.
Suppose now that:

$$
\begin{aligned}
\left(v_{l_{1}}, v_{l_{1}+1}, \ldots, v_{n}\right) & =\left(2,2, \ldots, 2, v_{l_{2}+1}\right. \\
\left(u_{l_{1}}, u_{l_{1}+1}, \ldots, u_{n}\right) & =(x, 0, \ldots, *) \\
, u_{l_{2}+1} & =g, *, \ldots, *)
\end{aligned}
$$

where $x \in\{1,0\}$ and $(f, g) \neq(2,0)$.
Then $P_{u, v}(q)=(1+q)^{l_{2}-l_{1}} P_{l_{2}}$ and $P_{l_{2}}$ is indexed by elements whose difference of the length is $\left(l(u, v)-2\left(l_{2}-l_{1}+1\right)+x\right)$. If $P_{u, v}(q)$ has degree $\frac{1}{2}(l(u, v)-1)$ then $P_{l_{2}}$ has degree $\frac{1}{2}\left(l(u, v)-1-2\left(l_{2}-l_{1}\right)\right)$. This happens if and only if $x=1$ and $P_{l_{2}}$ is indexed by equal elements.

Example 4 The Kazhdan-Lusztig polynomial indexed by $u=s_{1} s_{3} s_{7} s_{4} s_{3} s_{2}$ and
$v=s_{1} s_{3} s_{4} s_{5} s_{6} s_{7} s_{6} s_{5} s_{4} s_{3} s_{2}$ in $\mathfrak{S}(8)$ has the highest possible degree. In fact, the Boolean sequences $(1,1,2,1,0,0,1)$ and $(1,1,2,2,2,2,1)$ associated to $u$ and $v$ satisfy the requirement of Corollary 2.4.1 with $l_{1}=4$ and $l_{2}=6$.

The case of $(W, S)$ being generic linear Coxeter system is treated by the following theorems, whose proofs easily derive from Theorems 2.2.3 and 2.2.4.

Corollary 2.4.2 Under the hypotheses of Theorem 2.2.3, assume $l(u, v)>1$. Then $u \prec v$ if and only if $\psi(u) \prec \psi(v)$ in $(m+1)$.

Let $W$ be a non-strictly linear Coxeter system, $w \in W$ be a Boolean element, and $\bar{w}$ be a reduced expression of $w$ which is a subword of the Boolean expression $t_{1} \ldots t_{n-1} t_{n} t_{n-1} \ldots t_{1}$. We denote by $i_{L, t_{k}}(w)$ and $i_{R, t_{k}}(w)$ the elements represented by the expressions we obtain by inserting a factor $t_{k}$ to the left and to the right, respectively, in the appropriate position in $\bar{w}$. For instance, if $\bar{w}=t_{1} t_{3} t_{4} t_{2} t_{1}$, then $i_{L, t_{2}}(w)=t_{1} t_{2} t_{3} t_{4} t_{2} t_{1}$ and $i_{R, t_{3}}(w)=t_{1} t_{3} t_{4} t_{3} t_{2} t_{1}$.

Corollary 2.4.3 Under the hypotheses of Theorem 2.2.4, assume $l(u, v)>1$. Denote by $u^{\prime}$ and $v^{\prime}$ the elements represented by the expressions we obtain by deleting all the letters $t_{1}$ in $\bar{u}$ and $\bar{v}$. Then $u \prec v$ if and only if either

$$
-\bar{v}\left(t_{1}\right)=\bar{u}\left(t_{1}\right) \text { and } u^{\prime} \prec v^{\prime} \text {, or }
$$

- $\left(\bar{v}\left(t_{1}\right), \bar{u}\left(t_{1}\right), \bar{u}\left(t_{2}\right), \bar{u}\left(t_{j}\right)\right)=(2,1,0,0)$ and there exists $w \in\left\{i_{L, t_{2}}\left(u^{\prime}\right), i_{R, t_{2}}\left(u^{\prime}\right), i_{L, t_{j}}\left(u^{\prime}\right), i_{R, t_{j}}\left(u^{\prime}\right)\right\}$ such that $w \prec v^{\prime}$.


### 2.5 Kazhdan-Lusztig elements

Consider the basis $\mathcal{C}$ of the Hecke algebra $\mathcal{H}$ associated to a Coxeter system ( $W, S$ ) appearing in Theorems 0.5.4. In this section we compute those KazhdanLusztig elements which are indexed by Boolean elements in any linear Coxeter system. For any expression $\bar{x}=s_{i_{1}} \ldots s_{i_{r}}$, we set $C(\bar{x}):=C_{s_{i_{1}}} \ldots C_{s_{i_{r}}}$.
First we treat the case $W=\mathfrak{S}(n+1)$. If $\bar{x}$ is a subword of $s_{1} \ldots s_{n-1} s_{n} s_{n-1} \ldots s_{1}$ such that $\bar{x}\left(s_{k}\right)=2$ and $\bar{x}\left(s_{k+1}\right)=1$, we denote by $C^{k}(\bar{x})$ the element we obtain from $C(\bar{x})$ by deleting the factor $C_{s_{k+1}}$ and one of the two factors $C_{s_{k}}$ (by Proposition 0.5.5, it is easy to see that it does not matter which one). We extend this notation to $C^{K}(\bar{x})$, for any $K \subseteq[n]$, making the same deletions for every $k \in K$.

Theorem 2.5.1 Let $w \in \mathfrak{S}(n+1)$ be a Boolean element. Fix a reduced expression $\bar{w}$ of $w$ which is a subword of $s_{1} \ldots s_{n} \ldots s_{1}$ and let $V=\left\{k \in[n]: w_{k}=\right.$ $\left.2, w_{k+1}=1\right\}$. Then:

$$
C_{w}=\sum_{K \subseteq V}(-1)^{|K|} C^{K}(\bar{w})
$$

Proof. We use the recursive property of Proposition 0.5 .5 applied to $s_{1}$.
If $w_{1}=1$, and if we assume that the factor $s_{1}$ is on the left in $\bar{w}$, then $C_{w}=$ $C_{s_{1}} C_{s_{1} w}$ because $s_{1} \nless s_{1} w$.
If $w_{1}=2$, necessarily $w_{2} \neq 0$. Fix a reduced expression $\bar{z}$, which is a subword of $s_{1} \bar{w}$, for any element $z$ in $\left\{z \leq s_{1} w: s_{1} \in D_{L}(z)\right\}$. Then $\bar{z}$ has a factor $s_{1}$ on the right and, by Lemma 1.1.1, $\bar{z}\left(s_{2}\right)=0$. Hence, by Corollary 2.4.1, $\mu\left(z, s_{1} w\right) \neq 0$ if and only if $l\left(z, s_{1} w\right)=1$, that is to say if and only if $s_{2} z=s_{1} w$. This means that the sum is nonzero if and only if $w_{2}=1$, and, assuming that $\bar{w}$ has only one factor $s_{2}$ on the left, we have $C_{w}=C_{s_{1}} C_{s_{1} w}-C_{s_{2} s_{1} w}$. Applying the recursive property in its right version, we get:

$$
C_{w}=C_{s_{1}} C_{s_{1} w s_{1}} C_{s_{1}}-C_{s_{2} s_{1} w s_{1}} C_{s_{1}} .
$$

The result follows by iterating this procedure.
As a corollary, we have the following nice factorization.

Corollary 2.5.2 Let $w \in \mathfrak{S}(n+1)$ be a Boolean element. Fix a reduced expression $\bar{w}$ of $w$ which is a subword of $s_{1} \ldots s_{n} \ldots s_{1}$ and let $V^{\prime}=V+1=\{k \in$ $\left.[n]: w_{k-1}=2, w_{k}=1\right\}$. Then $C_{w}$ is obtained from $C(\bar{w})$ by changing the factor $C_{s_{k}}$ to $\left[C_{s_{k}}-\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right)^{-1} C_{e}\right]$ for all $k \in V^{\prime}$.

Proof. The assertion follows by the multiplication rule of Proposition 0.5.5.
Example 5 Let $w=s_{1} s_{2} s_{3} s_{5} s_{4} s_{3} s_{1} \in \mathfrak{S}(6)$. Then $V=\{1,3\}$ and
$C_{w}=C_{s_{1}} C_{s_{2}} C_{s_{3}} C_{s_{5}} C_{s_{4}} C_{s_{3}} C_{s_{1}}-C_{s_{3}} C_{s_{5}} C_{s_{4}} C_{s_{3}} C_{s_{1}}-C_{s_{1}} C_{s_{2}} C_{s_{3}} C_{s_{5}} C_{s_{1}}+C_{s_{3}} C_{s_{5}} C_{s_{1}}$,
while $V^{\prime}=\{2,4\}$ and we obtain the factorization:

$$
C_{w}=C_{s_{1}}\left[C_{s_{2}}-\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right)^{-1} C_{e}\right] C_{s_{3}} C_{s_{5}}\left[C_{s_{4}}-\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right)^{-1} C_{e}\right] C_{s_{3}} C_{s_{1}}
$$

Now we treat the case of a strictly linear Coxeter system $(W, S)$. Let $t \in T(W)$ be a Boolean reflection with Boolean expression $\bar{t}=t_{1} \ldots t_{n-1} t_{n} t_{n-1} \ldots t_{1}$ that
we can assume equal to

$$
s_{a} s_{a-1} \ldots s_{i+1} s_{b} s_{b+1} \ldots s_{i-1} s_{i} s_{i-1} \ldots s_{b+1} s_{b} s_{i+1} \ldots s_{a-1} s_{a}
$$

by Proposition 1.1.4. Suppose that $t_{j}$ is $s_{i+1}$. As before, if $\bar{x}$ is a subword of $t_{1} \ldots t_{n-1} t_{n} t_{n-1} \ldots t_{1}$ such that $\bar{x}\left(t_{k}\right)=2$ and $\bar{x}$ has only one factor $t_{k^{\prime}}, k^{\prime}>k$, that does not commute with $t_{k}\left(t_{k^{\prime}}=t_{k+1}\right.$, if $k \neq j, t_{k^{\prime}}=t_{n}$, if $\left.k=j\right)$, we denote by $C^{k}(\bar{x})$ the element we obtain from $C(\bar{x})$ by deleting the factor $C_{t_{k^{\prime}}}$ and one of the two factors $C_{t_{k}}$. We extend this notation to $C^{K}(\bar{x})$, for any $K \subseteq[n]$, making the same deletions for every $k \in K$. Keeping these notations, we have the following.

Theorem 2.5.3 Let $\left(W, S=\left\{s_{1}, \ldots, s_{m}\right\}\right)$ be a strictly linear Coxeter system, $w \in W, w \leq t$. Fix a reduced expression $\bar{w}$ of $w$ which is a subword of $t_{1} \ldots t_{n} \ldots t_{1}$, and let $V^{\prime}=\left\{k \in[n] \backslash\{j\}: \bar{w}\left(t_{k}\right)=2, \bar{w}\left(t_{k+1}\right)=1\right\}$ and

$$
V= \begin{cases}V^{\prime} \cup\{j\} & \text { if } \bar{w}\left(t_{j}\right)=2, \bar{w}\left(t_{n-1}\right) \neq 2, \\ V^{\prime} & \text { otherwise. }\end{cases}
$$

Then

$$
C_{w}=\sum_{K \subseteq V}(-1)^{|K|} C^{K}(\bar{w}) .
$$

Proof. The proof of Theorem 2.5.1 holds replacing $s_{1}$ with $t_{1}$, except when $t_{1}=t_{j}$. Let us treat this case. If $\bar{w}\left(t_{j}\right)=1$, and if we assume that the factor $t_{j}$ is on the left in $\bar{w}$, then $C_{w}=C_{t_{j}} C_{t_{j} w}$ because $t_{j} \not \leq t_{j} w$.
If $\bar{w}\left(t_{j}\right)=2$, necessarily $\bar{w}\left(t_{n}\right)=1$. Fix a reduced expression $\bar{z}$, which is a subword of $t_{j} \bar{w}$, for any element $z$ in $\left\{z \leq t_{j} w: t_{j} \in D_{L}(z)\right\}$. Then $\bar{z}$ has a factor $t_{j}$ on the right and, by Lemma 1.1.1, $\bar{z}\left(t_{n}\right)=0$. Hence, by Corollary 2.4.2, $\mu\left(z, t_{j} w\right) \neq 0$ if and only if $l\left(z, t_{j} w\right)=1$, that is to say if and only if $\bar{z}$ is obtain from $t_{j} \bar{w}$ by deleting the factor $t_{n}$. Such expression $\bar{z}$ would be reduced only if $\bar{w}\left(t_{n-1}\right) \neq 2$. In this case, we have $C_{w}=C_{t_{j}} C_{t_{j} w}-C_{z}$. Applying the recursive property in its right version, we get:

$$
C_{w}=C_{t_{j}} C_{t_{j} w t_{j}} C_{t_{j}}-C_{z t_{j}} C_{t_{j}}
$$

The assertion follows by iteration.
Theorem 2.5.4 Let $\left(W, S=\left\{s_{1}, \ldots, s_{m}\right\}\right)$ be a non-strictly linear Coxeter sys-
tem, $t \in T(W)$ be a Boolean reflection. Let $w \in W, w \leq t$ be such that $s_{i} \leq w$ for all $i \in[m]$. Fix a Boolean expression $\bar{t}=t_{1} \ldots t_{m} \ldots t_{1}$ of the type of Proposition 1.1.5 and a reduced expression $\bar{w}$ of $w$ which is a subword of $\bar{t}$. Then

$$
C_{w}=\left\{\begin{array}{lc}
C_{t_{1}} C_{w^{\prime}} C_{t_{1}}, & \text { if } \bar{w}\left(t_{1}\right)=2 \\
C_{t_{1}} C_{w^{\prime}}, & \text { if } \bar{w} \text { has only a factor } t_{1} \text { at the leftmost place }, \\
C_{w^{\prime}} C_{t_{1}}, & \text { if } \bar{w} \text { has only a factor } t_{1} \text { at the rightmost place },
\end{array}\right.
$$

where $w^{\prime}$ is the element represented by the expression we obtain from $\bar{w}$ by erasing all the factors $t_{1}$. Hence $C_{w^{\prime}}$ can be computed as in Theorem 2.5.3.

Proof. We use the recursive property of Proposition 0.5 .5 applied to $t_{1}$. If $\bar{w}\left(t_{1}\right)=1$, and if we assume that the factor $t_{1}$ is on the left, then $C_{w}=C_{t_{1}} C_{t_{1} w}$ because $t_{1} \nless t_{1} w$.
Let $\bar{w}\left(t_{1}\right)=2$. Suppose that $t_{j}$ is, together with $t_{2}$, the only other generator that does not commute with $t_{1}$. Fix a reduced expression $\bar{z}$, which is a subword of $\bar{t}_{1} w$, for any element $z$ in $\left\{z \leq t_{1} w: t_{1} \in D_{L}(z)\right\}$. Necessarily, $\bar{z}$ has a factor $t_{1}$ on the right and, by Lemma 1.1.1, $\bar{z}\left(t_{i}\right)=0$ for $i=2, j$. Hence $l\left(z, t_{1} w\right)>1$ and $\mu\left(z, t_{1} w\right) \neq 0$ by Corollary 2.4.3. So $C_{w}=C_{t_{1}} C_{t_{1} w}$. Applying the recursive property of Proposition 0.5.5 in its right version, we get

$$
C_{w}=C_{t_{1}} C_{t_{1} w t_{1}} C_{t_{1}}
$$

and the assertion is proved.

### 2.6 Poincaré polynomials

Given $v \in W$, define $F_{v}(q):=\sum_{u \leq v} q^{l(u)} P_{u, v}(q)$. It is known that, if $W$ is any Weyl or affine Weyl group, $F_{v}(q)$ is the intersection homology Poincaré polynomial of the Schubert variety indexed by $v$ (see [41]). In this section, we want to compute these polynomials when $W$ is a linear Coxeter system and $v \in W$ is a Boolean element.

First let us do this computation for $W=\mathfrak{S}(n+1)$, where we treat the Boolean elements in terms of $n$-Boolean sequences as in Section 1.2. Let us restrict the domain of $\phi$ to the interval $[e, v]$. Given any Boolean sequence
$u=\left(u_{i}, \ldots, u_{n}\right) \leq \phi(v)$ in the component-wise partial order, we define

$$
\begin{aligned}
n(u, \phi(v)) & :=\left|\left\{i \in[n-1]: v_{i}=2, u_{i}=1, u_{i+1} \neq 0\right\}\right| \\
b(u, \phi(v)) & :=\left|\left\{i \in[n-1]: v_{i}=2, v_{i+1}=2, u_{i+1}=0\right\}\right|
\end{aligned}
$$

With these notations,

$$
\phi_{\left.\right|_{[e, v]}}^{-1}(u)=2^{n(u, \phi(v))}
$$

and, by Theorem 2.2.1,

$$
F_{v}(q)=\sum_{u \leq \phi(v)} q^{l(u)}(1+q)^{b(u, \phi(v))} 2^{n(u, \phi(v))} .
$$

We have the following theorem.

Theorem 2.6.1 Let $v \in \mathfrak{S}(n+1)$ be a Boolean element. Then

$$
F_{v}(q)=(1+q)^{l(v)-2 f(v)}\left(1+q+q^{2}\right)^{f(v)},
$$

where $f(v)$ is the number of occurrences of the pattern $|2,1|$ in the sequence $\phi(v)$.

Proof. We proceed by induction on $l(v)$. When not specified, a sequence is meant to be Boolean, and we write $v$ instead of $\phi(v)$ to simplify notation.
We distinguish 2 cases.

1) $v_{1}=1$.

If we split the sum into two sums according as to whether $u_{1}=0$ or $u_{1}=1$, we obtain:

$$
F_{v}(q)=\sum_{u \leq v[2]} q^{l(u)}(1+q)^{b(u, v)} 2^{n(u, v)}+\sum_{u \leq v: u_{1}=1} q^{l(u)}(1+q)^{b(u, v)} 2^{n(u, v)}
$$

where, for all $i \in[n]$,

$$
v[i]_{j}=\left\{\begin{array}{lc}
v_{j}, & \text { if } j \geq i \\
0, & \text { otherwise }
\end{array}\right.
$$

Note that if $v$ is Boolean, so is $v[i]$ for all $i$.
Clearly $b(u, v)=b(u, v[2])$ and $n(u, v)=n(u, v[2])$. Sending $u$ to $u[2]$, we obtain a bijection between the sequences $u \leq v$ such that $u_{1}=1$ and the sequences $u \leq$ $v[2]$. Since $l(u)=l(u[2])+1, b(u, v)=b(u[2], v[2])$ and $n(u, v)=n(u[2], v[2])$,
we get:

$$
F_{v}(q)=\sum_{u \leq v[2]} q^{l(u)}(1+q)^{b(u, v[2])} 2^{n(u, v[2])}+\sum_{u \leq v[2]} q^{l(u)+1}(1+q)^{b(u, v[2])} 2^{n(u, v[2])},
$$

that is $F_{v}(q)=(1+q) F_{v[2]}(q)$, and we conclude by induction.
2) $v_{1}=2$.

Splitting the sum, we get:
$F_{v}(q)=\sum_{u \leq v: u_{1} \neq 2} q^{l(u)}(1+q)^{b(u, v)} 2^{n(u, v)}+\sum_{u \leq v: u_{1}=2} q^{l(u)}(1+q)^{b(u, v)} 2^{n(u, v)}$.
Being $u_{1} \neq 2$, the first sum is over all the sequences $u \leq v^{\prime}$, where

$$
v_{j}^{\prime}= \begin{cases}v_{j}, & \text { if } j \neq 1, \\ 1, & \text { if } j=1,\end{cases}
$$

and

$$
\begin{aligned}
& b(u, v)=\left\{\begin{array}{lc}
b\left(u, v^{\prime}\right)+1, & \text { if } v_{2}=2 \text { and } u_{2}=0, \\
b\left(u, v^{\prime}\right), & \text { otherwise },
\end{array}\right. \\
& n(u, v)=\left\{\begin{array}{lr}
n\left(u, v^{\prime}\right)+1, & \text { if } u_{1}=1 \text { and } u_{2} \neq 0, \\
n\left(u, v^{\prime}\right), & \text { otherwise. }
\end{array}\right.
\end{aligned}
$$

As to the second sum, there is a bijection between the sequences $u \leq v$ such that $u_{1}=2$ and the sequences $u \leq v^{\prime}$ such that $u_{1}=1$ and $u_{2} \neq 0$. This bijection sends $u$ to $u^{\prime}$ (similar definition as for $v^{\prime}$ ). Clearly $l(u)=l\left(u^{\prime}\right)+1$, $b(u, v)=b\left(u^{\prime}, v^{\prime}\right)$ and $n(u, v)=n\left(u^{\prime}, v^{\prime}\right)$.
Then, if $v_{2}=2$, combining all these facts we obtain:

$$
\begin{aligned}
& F_{v}(q)=(1+q) \sum_{u \leq v^{\prime}: u_{2}=0} q^{l(u)}(1+q)^{b\left(u, v^{\prime}\right)} 2^{n\left(u, v^{\prime}\right)} \\
&+2 \sum_{u \leq v^{\prime}:}: u_{1}=1 u_{2} \neq 0 \\
& q^{l(u)}(1+q)^{b\left(u, v^{\prime}\right)} 2^{n\left(u, v^{\prime}\right)} \\
&+\sum_{u \leq v^{\prime}:} \sum_{u_{1}=0} q^{l(u)}(1+q)^{b\left(u, v^{\prime}\right)} 2^{n\left(u, v^{\prime}\right)} \\
&+q \sum_{u \leq v^{\prime}:}: u_{1}=1 u_{2} \neq 0
\end{aligned} q^{l(u)}(1+q)^{b\left(u, v^{\prime}\right)} 2^{n\left(u, v^{\prime}\right)} .
$$

By an easy bijection sending $u$ to $u[2]$,
$\sum_{u \leq v^{\prime}: u_{1}=1: u_{2} \neq 0} q^{l(u)}(1+q)^{b\left(u, v^{\prime}\right)} 2^{n\left(u, v^{\prime}\right)}=q \sum_{u \leq v^{\prime}: u_{1}=0} q^{l(u)}(1+q)^{b\left(u, v^{\prime}\right)} 2^{n\left(u, v^{\prime}\right)}$
and hence we obtain $F_{v}(q)=(1+q) F_{v^{\prime}}(q)=(1+q)^{2} F_{v[2]}$, where the last equality follows by case 1 ). So we conclude by induction.
On the other hand, if $v_{2}=1$, we obtain

$$
F_{v}(q)=(1+q)^{2} F_{v[2]}-q \sum_{u \leq v^{\prime}: u_{2}=0} q^{l(u)}(1+q)^{b\left(u, v^{\prime}\right)} 2^{n\left(u, v^{\prime}\right)}
$$

Now, by case 1$), F_{v[2]}=(1+q) F_{v[3]}$, while

$$
\sum_{u \leq v^{\prime}: u_{2}=0} q^{l(u)}(1+q)^{b\left(u, v^{\prime}\right)} 2^{n\left(u, v^{\prime}\right)}=\sum_{u \leq v^{\prime}: u_{1}=0: u_{2}=0} q^{l(u)}(1+q)^{b\left(u, v^{\prime}\right)} 2^{n\left(u, v^{\prime}\right)}
$$

which is equal to $F_{v[3]}+q F_{v[3]}$.
Hence:

$$
F_{v}(q)=(1+q)^{3} F_{v[3]}-\left(q+q^{2}\right) F_{v[3]}(q)=(1+q)\left(1+q+q^{2}\right) F_{v[3]},
$$

and we conclude by induction.

Example 6 Let $v \in \mathfrak{S}(8)$, $v=s_{1} s_{2} s_{4} s_{5} s_{6} s_{7} s_{5} s_{4} s_{3} s_{2}$. Then the Boolean sequence associated to $v$ is $(1,2,1,2,2,1,1), f(v)=2$ and $F_{v}(q)=(1+q)^{l(v)-4}(1+$ $\left.q+q^{2}\right)^{2}$.

The following two theorems treat respectively the case of a strictly and of a non-strictly linear Coxeter system.

Theorem 2.6.2 Let $\left(W, S=\left\{s_{1}, \ldots, s_{m}\right\}\right)$ be a strictly linear Coxeter system, $t \in W$ a Boolean reflection and $v \in W, v \leq t$. Then $F_{v}(q)=F_{\psi(v)}(q)$, where $\psi$ is as in Lemma 2.2.2 and $F_{\psi(v)}(q)$ can be computed as in Theorem 2.6.1.

Proof. Clear since $\psi:[e, v]_{W} \rightarrow[e, \psi(v)]_{\mathfrak{S}(m+1)}$ is an isomorphism of posets
preserving the length and $P_{u, v}(q)=P_{\psi(u), \psi(v)}(q)$ for all $u \in[e, v]_{W}$ by Theorem 2.2.3.

Theorem 2.6.3 Let $\left(W, S=\left\{s_{1}, \ldots, s_{m}\right\}\right)$ be a non-strictly linear Coxeter system, $t \in T(W)$ a Boolean reflection that we can assume such that $s_{i} \leq t$ for all $i \in[m]$, and $v \in W, v \leq t$. Fix a Boolean expression $\bar{t}=t_{1} \ldots t_{n-1} t_{n} t_{n-1} \ldots t_{1}$ of $t$ of the type shown in Proposition 1.1.5 and a reduced expression $\bar{v}$ of $v$ which is a subword of $\bar{t}$. Then $F_{v}(q)=(1+q)^{\bar{v}\left(t_{1}\right)} F_{v^{\prime}}(q)$, where $v^{\prime}$ is the element of $W$ represented by the expression we obtain from $\bar{v}$ by deleting all the letters $t_{1}$ and $F_{v^{\prime}}(q)$ can be computed as in Theorem 2.6.2.

Proof. Suppose that $t_{j}$ is, together with $t_{2}$, the only other generator that does not commute with $t_{1}$, and fix, for any element $u \leq v$, an expression $\bar{u}$ of $u$ which is a subword of $\bar{v}$. Let us denote by $u^{\prime}$ the element represented by the expression we obtain from $\bar{u}$ by deleting all the letters $t_{1}$. We distinguish 2 cases.

1) $\bar{v}\left(t_{1}\right)=1$.

If we split the sum into two sums, by Theorem 2.2.4, we obtain:

$$
F_{v}(q)=\sum_{\bar{u}\left(t_{1}\right)=1} q^{l(u)} P_{u^{\prime}, v^{\prime}}+\sum_{\bar{u}\left(t_{1}\right)=0} q^{l(u)} P_{u, v^{\prime}}
$$

Since in the first sum $l(u)=l\left(u^{\prime}\right)+1$ and since there is a bijection between the two sets over which we are summing, we get:

$$
F_{v}(q)=(1+q) F_{v^{\prime}}(q) .
$$

2) $\bar{v}\left(t_{1}\right)=2$

Splitting the sum, we obtain:

$$
F_{v}(q)=\sum_{\bar{u}\left(t_{1}\right)=2} q^{l(u)} P_{u, v}+\sum_{\bar{u}\left(t_{1}\right)=1} q^{l(u)} P_{u, v}+\sum_{\bar{u}\left(t_{1}\right)=0} q^{l(u)} P_{u, v} .
$$

After some simplifications by means of Theorem 2.2.4 and of natural maps, the first sum gets equal to:

$$
\sum_{\left(\bar{u}^{\prime}\left(t_{2}\right), \bar{u}^{\prime}\left(t_{j}\right)\right) \neq(0,0)} q^{l\left(u^{\prime}\right)+2} P_{u^{\prime}, v^{\prime}}
$$

the second to:

$$
2 \sum_{\left(\bar{u}^{\prime}\left(t_{2}\right), \bar{u}^{\prime}\left(t_{j}\right)\right) \neq(0,0)} q^{l\left(u^{\prime}\right)+1} P_{u^{\prime}, v^{\prime}}+(1+q) \sum_{\left(\bar{u}^{\prime}\left(t_{2}\right), \bar{u}^{\prime}\left(t_{j}\right)\right)=(0,0)} q^{l\left(u^{\prime}\right)+1} P_{u^{\prime}, v^{\prime}},
$$

(the " 2 " comes out from the fact that the map is 2 to 1 ), the third to

$$
\sum_{\left(\bar{u}^{\prime}\left(t_{2}\right), \bar{u}^{\prime}\left(t_{j}\right)\right) \neq(0,0)} q^{l\left(u^{\prime}\right)} P_{u^{\prime}, v^{\prime}}+(1+q) \sum_{\left(\bar{u}^{\prime}\left(t_{2}\right), \bar{u}^{\prime}\left(t_{j}\right)\right)=(0,0)} q^{l\left(u^{\prime}\right)} P_{u^{\prime}, v^{\prime}} .
$$

By adding the summands, we finally obtain:

$$
F_{v}(q)=(1+q)^{2} F_{v^{\prime}}
$$

and the assertion is proved.

Remark. The polynomials $F_{v}(q)$ computed in this section are all symmetric and unimodal. For Weyl or affine Weyl groups $W$, this is a consequence of the fact that (middle perversity) intersection cohomology satisfies Poincaré duality and the "Hard Lefschetz Theorem". So this result is consistent with the idea that there may be geometric objects associated to any Coxeter group analogous to Schubert varieties.

## Chapter 3

## Parabolic $R$-polynomials and Kazhdan-Lusztig polynomials


#### Abstract

This chapter deals concretely with the computation of the parabolic analogues of the Kazhdan-Lusztig and $R$-polynomials for the symmetric group. We give closed combinatorial product formulae for the parabolic $R$-polynomials of both types $q$ and -1 , and for the parabolic Kazhdan-Lusztig polynomials of type $q$. These formulae are valid in the case that the indexing permutations are Boolean, and with no restrictions on the parabolic subgroup $W_{J}$. These parabolic KazhdanLusztig and $R$-polynomials turn out to depend on the number of occurrences of certain sub-tableaux in a fixed tableau associated to the indexing permutations. Throughout this chapter, we make use of the notion of the maps $\phi_{R}(u, v)$ and $\phi_{L}(u, v)$ we introduced in Section 1.2.


### 3.1 Parabolic $R$-polynomials

Let $u, v \in \mathfrak{S}(n+1)^{J}, u \leq v$, be two Boolean permutations. In this section we give a closed combinatorial formula for the parabolic $R$-polynomials of both types $q$ and -1 indexed by $u$ and $v$. In this formula, there are no restrictions on the subset $J$ of $S$.

Let $(\bar{u}, \bar{v})$ be the right Boolean expressions of $(u, v)$ and consider $\phi_{R}(u, v)$. First we need the following proposition.

Proposition 3.1.1 Suppose that $u, v \leq s_{1} s_{2} \cdots s_{n}$. Then

$$
R_{u, v}^{J, q}(q)=(q-1)^{l(u, v)-E(u, v)}(q-1-x)^{E(u, v)}
$$

where

$$
E(u, v)=\left\lvert\, \begin{gathered}
\left.\begin{array}{c}
\circ \\
* \\
\hline 0 \mid \\
\hline 0 \\
\hline
\end{array} \right\rvert\, . \\
\hline
\end{gathered} .\right.
$$

Proof. We proceed by induction on $n$, the case $n=1$ being clear. If $v \leq$ $s_{1} s_{2} \cdots s_{n-1}$, then we conclude by induction.
So assume that $s_{n}$ is the rightmost letter of $\bar{v}$ (equivalently assume that $v_{n}=1_{l}$ ). Apply 3) of Theorem 0.5 .11 to $s_{n}$. If $u_{n}=1_{l}$, then $R_{u, v}^{J, q}(q)=R_{u s_{n}, v s_{n}}^{J, q}(q)$ and we conclude by induction. If $u_{n}=0$, then $s \notin D_{R}(u)$. By Table 1.2, $u s_{n} \notin \mathfrak{S}(n+1)^{J}$ if and only if $n \in J$ and $u_{n-1}=0$. In this case $R_{u, v}^{J, q}(q)=$ $(q-1-x) R_{u, v s_{n}}^{J, q}(q)$. Otherwise, $u s_{n} \in \mathfrak{S}(n+1)^{J}$ and

$$
R_{u, v}^{J, q}(q)=(q-1) R_{u, v s_{n}}^{J, q}(q)+q R_{u s_{n}, v s_{n}}^{J, q}(q) .
$$

But $u s_{n} \not \leq v s_{n}$ because $s_{n} \leq u s_{n}$ and $s_{n} \not \leq v s_{n}$. So $R_{u, v}^{J, q}(q)=(q-1) R_{u, v s_{n}}^{J, q}(q)$ and the assertion follows by induction.
Note that Proposition 3.1.1, which is stated for the symmetric group, can be easily generalized to any Coxeter group $W$.

Now we want to associate to $(u, v)$ the pair of elements we obtain from the right Boolean expressions $(\bar{u}, \bar{v})$ of $(u, v)$ by deleting all the letters on the right. Precisely, we define a map $\theta:\{(x, y) \in[e,(1, n+1)] \times[e,(1, n+1)]: x \leq y\} \rightarrow$ $\left\{(x, y) \in\left[e, s_{1} \cdots s_{n}\right] \times\left[e, s_{1} \cdots s_{n}\right]: x \leq y\right\}$ as follows. Given $(x, y)$ in the range, we obtain $\phi_{R}(\theta(x, y))$ from $\phi_{R}(x, y)$ by changing all 2 to $1_{l}$ and all $1_{r}$ to 0 . In particular, $\theta$ does not depend on $J$.
For example, if
then

$$
\begin{aligned}
& \times 0 \bigcirc \times 0 \times \times \times \times \times
\end{aligned}
$$

For $m \geq 1$, let

$$
\begin{aligned}
A_{m}(u, v):=\mid\{i \in J: & {[i+1, i+m-1] \subseteq J, } \\
& \left(v_{i}, v_{i+1}, \ldots, v_{i+m-1}\right)=(2,2, \ldots, 2), \\
& \left(u_{i-1}, u_{i}, \ldots, u_{i+m}\right)=\left(1_{l}, 0, \ldots, 0\right) \text { and } \\
& \text { either } \left.i+m \notin J \text { or }\left(v_{i+m}, u_{i+m+1}\right)=\left(1_{r}, \neq 0\right)\right\} \mid+ \\
\mid\{i \notin J: & {[i+1, i+m-1] \subseteq J, } \\
& \left(v_{i}, v_{i+1}, \ldots, v_{i+m-1}\right)=(2,2, \ldots, 2), \\
& \left(u_{i}, u_{i+1}, \ldots, u_{i+m}\right)=(0,0, \ldots, 0) \text { and } \\
& \text { either } \left.i+m \notin J \text { or }\left(v_{i+m}, u_{i+m+1}\right)=\left(1_{r}, \neq 0\right)\right\} \mid .
\end{aligned}
$$

Equivalently,

| $\circ$ |
| :---: |
| 2 |
| 0 |

where the columns of type 0 are exactly $m$ in the first two tableaux, $m-1$ in the other two.

Furthermore let
and finally

$$
M_{u, v}(q):=\prod_{m=1}^{\infty}\left[\frac{(-x)^{m+1}}{q-1}+(q-1-x)^{m}\right]^{A_{m}}
$$

Then the polynomial $R_{u, v}^{J, x}(q)$ can be computed through the following product formula. For notational convenience, here we drop the dependence on $(u, v)$ so that $l:=l(u, v), l(\theta):=l(\theta(u, v)), A_{m}:=A_{m}(u, v), B:=B(u, v)$, and $C:=C(u, v)$.

Theorem 3.1.2 The parabolic $R$-polynomial $R_{u, v}^{J, x}(q)$ is equal to

$$
\begin{equation*}
(q-1)^{B}(q-1-x)^{l-l(\theta)-\sum m A_{m}-B} M_{u, v}(q) R_{\theta(u, v)}^{J, x}(q), \tag{3.1}
\end{equation*}
$$

where the polynomial $R_{\theta(u, v)}^{J, x}(q)$ can be computed using the formula in Proposition 3.1.1.
Equivalently, $R_{u, v}^{J, x}(q)$ is equal to

$$
\begin{equation*}
(q-1)^{B+l(\theta)-C}(q-1-x)^{l-l(\theta)-\sum m A_{m}-B+C} M_{u, v}(q) . \tag{3.2}
\end{equation*}
$$

Proof. Throughout this proof we use Tits' Word Theorem (Theorem 0.3.6) as well as Lemma 1.1.2 without explicit mention.

Recall that $(\bar{u}, \bar{v})$ are the right Boolean expressions of $(u, v)$. First of all, the equivalence of 3.1 and 3.2 follows by Proposition 3.1.1. In fact $R_{\theta(u, v)}^{J, x}(q)$ has only factors $(q-1-x)$ and $(q-1)$, and $C(u, v)$ counts the sub-tableaux of $\stackrel{\circ}{*}{ }^{-} 1_{l}$ $\phi_{R}(u, v)$ that give rise to sub-tableaux of type 000 in $\phi_{R}(\theta(u, v))$.

Let us prove 3.1 by induction on $l(v)$. If $v \leq s_{1} s_{2} \cdots s_{n}$, we are done because $\theta(u, v)=(u, v), B(u, v)=0$ and $A_{m}(u, v)=0$ for all $m \geq 1$.
So we may assume that $v \notin s_{1} s_{2} \cdots s_{n}$. Let $s_{i}$ be the letter at the rightmost place in $\bar{v}$ and use the recursive property of Theorem 0.5 .11 applied to $s_{i}$. Case by case, we investigate the relationship between the polynomial $R_{u, v}^{J, x}(q)$ and the polynomial $R_{u^{\prime}, v^{\prime}}^{J, x}(q)$, where $u^{\prime}$ and $v^{\prime}$ are the elements represented by $\bar{u}$ and $\bar{v}$ with the letters $s_{i}$ at the rightmost place (if any) deleted. So $v^{\prime}=v s_{i}$ and $u^{\prime}=u$ or $u s_{i}$.
Let us collect the cases that are analogous.
If both $\bar{v}$ and $\bar{u}$ have a letter $s_{i}$ at the rightmost place, then $R_{u, v}^{J, x}(q)=R_{u^{\prime}, v^{\prime}}^{J, x}(q)$, and we conclude by induction.
Using Table 1.2, it is not hard to check that $u \leq u s_{i} \in W^{J}$ and $u s_{i} \not \leq v s_{i}$ precisely in the cases given in the following table, where empty space stands for any entry.

| $v_{i-1}$ | $v_{i}$ | $v_{i+1}$ | $v_{i+2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $u_{i-1}$ |  |  |  |$|$| $u_{i}$ | $u_{i+1}$ | $u_{i+2}$ |
| :---: | :---: | :---: |

In all these cases we have $R_{u, v}^{J, x}(q)=(q-1) R_{u^{\prime}, v^{\prime}}^{J, x}(q)$, while $B(u, v)=B\left(u^{\prime}, v^{\prime}\right)+1$ and $A_{m}(u, v)=A_{m}\left(u^{\prime}, v^{\prime}\right)$ for all $m \geq 1$. Hence the result follows by induction.

Similarly, $u \leq u s_{i} \notin W^{J}$ precisely in the cases given in the following table

| $\begin{gathered} v_{i-1} \\ u_{i-1} \end{gathered}$ | $v_{i}$ | $\begin{aligned} & v_{i+1} \\ & u_{i+1} \end{aligned}$ | $\begin{gathered} v_{i+2} \\ u_{i+2} \end{gathered}$ | $i \in J$ | $i+1 \in J$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | $1_{r}$ 0 | 0 |  | yes |  |
| 0 | 2 0 | 0 |  | yes |  |
|  | $\begin{gathered} 2 \\ 1_{l} \end{gathered}$ | $1_{l}$ |  | no | yes |
|  | $\begin{gathered} 2 \\ 1_{l} \end{gathered}$ | $1_{r}$ | 0 | no | yes |
| $1_{l}$ | $\begin{gathered} 2 \\ 1_{l} \end{gathered}$ | $1_{l}$ |  | yes | yes |
| $1_{l}$ | 2 1 1 | $1_{r}$ | 0 | yes | yes |

where $R_{u, v}^{J, x}(q)=(q-1-x) R_{u^{\prime}, v^{\prime}}^{J, x}(q)$, while $B(u, v)=B\left(u^{\prime}, v^{\prime}\right)$ and $A_{m}(u, v)=$ $A_{m}\left(u^{\prime}, v^{\prime}\right)$ for all $m \geq 1$. So the result follows by induction.

By Table1.2, we have that $u \leq u s_{i} \in W^{J}$ and $u s_{i} \leq v s_{i}$ exactly in the following two cases.

Case i) For some $m \geq 1,\left(v_{i}, v_{i+1}, \ldots, v_{i+m-1}\right)=(2,2, \ldots, 2),\left(u_{i-1}, u_{i}, \ldots, u_{i+m}\right)=$ $\left(1_{l}, 0, \ldots, 0\right),[i, i+m-1] \subset J$ and either $i+m \notin J$ or, if $i+m \in J,\left(v_{i+m}, u_{i+m+1}\right) \neq$ $(2,0)$.
First of all, let us treat the case $m=1$. By Theorem 0.5 .11 we have

$$
R_{u, v}^{J, x}(q)=(q-1) R_{u, v s_{i}}^{J, x}(q)+q R_{u s_{i}, v s_{i}}^{J, x}(q) .
$$

As $l\left(v s_{i}\right)=l(v)-1$, we can use the induction hypothesis and find
$R_{u, v s_{i}}^{J, x}(q)= \begin{cases}(q-1) R_{u s_{i}, v s_{i}}^{J, x}(q), & \text { if } i+1 \notin J, \\ (q-1) R_{u s_{i}, v s_{i}}^{J, x}(q), & \text { if } i+1 \in J \text { and } v_{i+1}=1_{r}, u_{i+2} \neq 0, \\ (q-1-x) R_{u s_{i}, v s_{i}}^{J, x}(q), & \text { if } i+1 \in J \text { and } v_{i+1} \in\left\{2,1_{l}\right\}, \\ (q-1-x) R_{u s_{i}, v s_{i}}^{J, x}(q), & \text { if } i+1 \in J \text { and }\left(v_{i+1}, u_{i+2}\right)=\left(1_{r}, 0\right),\end{cases}$
and hence
$R_{u, v}^{J, x}(q)= \begin{cases}\left(\frac{q^{2}-q+1}{q-1}\right) R_{u, v s_{i}}^{J, x}(q), & \text { if } i+1 \notin J, \\ \left(\frac{q^{2}-q+1}{q-1}\right) R_{u, v s_{i}}^{J, x}(q), & \text { if } i+1 \in J \text { and } v_{i+1}=1_{r}, u_{i+2} \neq 0, \\ (q-1-x) R_{u, v s_{i}}^{J, x}(q), & \text { if } i+1 \in J \text { and } v_{i+1} \in\left\{2,1_{l}\right\}, \\ (q-1-x) R_{u, v s_{i}}^{J, x}(q), & \text { if } i+1 \in J \text { and }\left(v_{i+1}, u_{i+2}\right)=\left(1_{r}, 0\right),\end{cases}$
(note that $(q-1)+\frac{q}{q-1-x}=(q-1-x)$ for $x \in\{-1, q\}$ ).
Now, for all $m \geq 1$, we want to investigate the relationship between $R_{u, v}^{J, x}(q)$ and $R_{u, v s_{i} \cdots s_{i+m-1}}^{J, x}(q)$. A priori, $R_{u, v}^{J, x}(q) / R_{u, v s_{i} \cdots s_{i+m-1}}^{J, x}(q)$ could be function of all the entries in $\phi_{R}(u, v)$ and we abuse notation by setting

$$
f(m)=\frac{R_{u, v}^{J, x}(q)}{R_{u, v s_{i} \cdots s_{i+m-1}}^{J, x}(q)}
$$

We claim that $f(m)$ only depends on $m, v_{i+m}, u_{i+m+1}$ and on whether $i+m$ is in $J$ or not. We prove the claim by induction on $m$. The claim is true for
$m=1$ since we have just proved that

$$
f(1)= \begin{cases}\left(\frac{q^{2}-q+1}{q-1}\right), & \text { if } i+1 \notin J  \tag{3.3}\\ \left(\frac{q^{2}-q+1}{q-1}\right), & \text { if } i+1 \in J \text { and } v_{i+1}=1_{r}, u_{i+2} \neq 0 \\ (q-1-x), & \text { if } i+1 \in J \text { and } v_{i+1} \in\left\{2,1_{l}\right\} \\ (q-1-x), & \text { if } i+1 \in J \text { and }\left(v_{i+1}, u_{i+2}\right)=\left(1_{r}, 0\right)\end{cases}
$$

If $m>1$, by Theorem 0.5.11,

$$
R_{u, v}^{J, x}(q)=(q-1) R_{u, v s_{i}}^{J, x}(q)+q R_{u s_{i}, v s_{i}}^{J, x}(q)
$$

by the induction hypothesis on 3.1

$$
R_{u, v s_{i}}^{J, x}(q)=(q-1-x)^{m-1} R_{u, v s_{i} \cdots s_{i+m-1}}^{J, x}(q)
$$

and by the induction hypothesis on the claim we can write

$$
R_{u s_{i}, v s_{i}}^{J, x}(q)=f(m-1) R_{u s_{i}, v s_{i} \cdots s_{i+m-1}}^{J, x}(q)
$$

By induction hypothesis on $3.1, R_{u, v s_{i} \cdots s_{i+m-1}}^{J, x}(q)=(q-1-x) R_{u s_{i}, v s_{i} \cdots s_{i+m-1}}^{J, x}(q)$, and hence $f(m)$ satisfies the following recursive property

$$
\begin{equation*}
f(m)=(q-1)(q-1-x)^{m-1}+\frac{q}{q-1-x} f(m-1) \tag{3.4}
\end{equation*}
$$

for any choice of $v_{i+m}, u_{i+m+1}$ and $J$. This prove the claim.
Now we can conclude that
$f(m)= \begin{cases}\frac{(-x)^{m+1}}{q-1}+(q-1-x)^{m}, & \text { if } i+m \notin J, \\ \frac{\left(-x^{m+1}\right.}{q-1}+(q-1-x)^{m}, & \text { if } i+m \in J \text { and } v_{i+m}=1_{r}, u_{i+m+1} \neq 0, \\ (q-1-x)^{m}, & \text { if } i+m \in J \text { and } v_{i+m} \in\left\{2,1_{l}\right\}, \\ (q-1-x)^{m}, & \text { if } i+m \in J \text { and }\left(v_{i+m}, u_{i+m+1}\right)=\left(1_{r}, 0\right) .\end{cases}$
In fact, for $x \in\{-1, x\}$, this function verifies both the recursive property of 3.4 and the initial conditions of 3.3.
Hence the result follows by induction.

Case ii) For some $m \geq 1,\left(v_{i}, v_{i+1}, \ldots, v_{i+m-1}\right)=(2,2, \ldots, 2),\left(u_{i}, u_{i+1}, \ldots, u_{i+m}\right)=$ $(0,0, \ldots, 0), i \notin J,[i+1, i+m-1] \subset J$ and either $i+m \notin J$ or, if $i+m \in J$,
$\left(v_{i+m}, u_{i+m+1}\right) \neq(2,0)$.
As in Case i), we can show by induction that $R_{u, v}^{J, x}(q) / R_{u, v s_{i} \cdots s_{i+m-1}}^{J, x}(q)$ only depends on $m, v_{i+m}, u_{i+m+1}$ and on whether $i+m$ is in $J$ or not. We abuse notation by setting

$$
g(m)=\frac{R_{u, v}^{J, x}(q)}{R_{u, v s_{i} \cdots s_{i+m-1}}^{J, x}(q)} .
$$

By Theorem 0.5.11 and the induction hypothesis on 3.1, we have

$$
\begin{aligned}
R_{u, v}^{J, x}(q)= & (q-1) R_{u, v s_{i}}^{J, x}(q)+q R_{u s_{i}, v s_{i}}^{J, x}(q) \\
= & (q-1)(q-1-x)^{m-1} R_{u, v s_{i} \cdots s_{i+m-1}}^{J, x}(q) \\
& +q f(m-1) R_{u s_{i}, v s_{i} \cdots s_{i+m-1}}^{J, x}(q)
\end{aligned}
$$

where $f(m)$ is as above. Now, by induction hypothesis on 3.1,

$$
R_{u, v s_{i} \cdots s_{i+m-1}}^{J, x}(q)=(q-1-x) R_{u s_{i}, v s_{i} \cdots s_{i+m-1}}^{J, x}(q)
$$

and hence we have

$$
\begin{equation*}
g(m)=(q-1)(q-1-x)^{m-1}+\frac{q}{q-1-x} f(m-1) . \tag{3.5}
\end{equation*}
$$

for any choice of $v_{i+m}, u_{i+m+1}$ and $J$.
We claim that $g(m)=f(m)$ for all $m$. By 3.4 and 3.5 it suffices to prove that $g(1)=f(1)$.
So assume $m=1$. By Theorem 0.5.11, we have

$$
R_{u, v}^{J, x}(q)=(q-1) R_{u, v s_{i}}^{J, x}(q)+q R_{u s_{i}, v s_{i}}^{J, x}(q)
$$

and by induction we have

$$
R_{u, v s_{i}}^{J, x}(q)= \begin{cases}(q-1) R_{u s_{i}, v s_{i}}^{J, x}(q), & \text { if } i+1 \notin J, \\ (q-1) R_{u s_{i}, v s_{i}}^{J, n}(q), & \text { if } i+1 \in J \text { and } v_{i+1}=1_{r}, u_{i+2} \neq 0, \\ (q-1-x) R_{u s_{i}, v s_{i}}^{J}(q), & \text { if } i+1 \in J \text { and } v_{i+1} \in\left\{2,1_{l}\right\}, \\ (q-1-x) R_{u s_{i}, v s_{i}}^{J, 2}(q), & \text { if } i+1 \in J \text { and }\left(v_{i+1}, u_{i+2}\right)=\left(1_{r}, 0\right),\end{cases}
$$

obtaining the same values of Case i). So $g(1)=f(1)$ and $g(m)=f(m)$ for all $m \geq 1$.

This concludes the induction step and hence the proof.

Example 7 Let us compute the $R$-polynomial $R_{u, v}^{J, x}(q)$ of $S_{12}$, where the Boolean permutations $v$ and $u$, and the subset $J$ of $S$ are as follows:

$$
\begin{aligned}
v & =s_{1} s_{2} s_{3} s_{4} s_{5} s_{6} s_{8} s_{9} s_{11} s_{10} s_{9} s_{8} s_{7} s_{6} s_{4} s_{3} s_{2} \\
u & =s_{1} s_{6} s_{11} s_{7} \\
J & =\{2,3,4,9,10\}
\end{aligned}
$$

By Table 1.2, the permutations $u$ and $v$ are in $S_{12}^{J}$. As the given expressions are right Boolean, we have
and

Now $l(u, v)=13, l(\theta(u, v))=6$ and

$$
\begin{aligned}
A_{2}(u, v) & =|\{8\}|=1 \\
A_{3}(u, v) & =|\{2\}|=1 \\
A_{m}(u, v) & =0 \text { for all } m \notin\{2,3\} .
\end{aligned}
$$

Hence

$$
M_{u, v}(q)=\left[\frac{(-x)^{3}}{q-1}+(q-1-x)^{2}\right]\left[\frac{(-x)^{4}}{q-1}+(q-1-x)^{3}\right] .
$$

Furthermore

$$
B(u, v)=2
$$

with the contributions exactly given by

If we want to use 3.1, we have to compute $R_{\theta(u, v)}^{J, x}(q)$. By Proposition 3.1.1,

$$
R_{\theta(u, v)}^{J, x}(q)=(q-1)^{3}(q-1-x)^{3},
$$

since $E(\theta(u, v))=3$.
If we want to use 3.2, we have to compute $C(u, v)$ and we obtain

$$
C(u, v)=3
$$

with the contributions exactly given by

$$
=3 .
$$

Note that $E(\theta(u, v))=C(u, v)$. This is not by chance.
Using one of the two equivalent formulae 3.1 and 3.2 we obtain
$R_{u, v}^{J, x}(q)=(q-1)^{5}(q-1-x)^{3}\left[\frac{(-x)^{3}}{q-1}+(q-1-x)^{2}\right]\left[\frac{(-x)^{4}}{q-1}+(q-1-x)^{3}\right]$,
that is

$$
R_{u, v}^{J, x}(q)= \begin{cases}q^{3}(q-1)^{3}\left(q^{3}-q^{2}+1\right)\left(q^{4}-q^{3}+1\right), & \text { if } x=-1 \\ (q-1)^{3}\left(q^{3}-q+1\right)\left(q^{4}-q+1\right), & \text { if } x=q\end{cases}
$$

## Remarks.

- Theorem 3.1.2, as stated, fails for the left Boolean expressions.
- The result in Theorem 3.1.2 for $J=\emptyset$ (ordinary $R$-polynomials) implies Theorem 2.1.1.


### 3.2 Parabolic Kazhdan-Lusztig polynomials

Let $u, v \in \mathfrak{S}(n+1)^{J}, u \leq v$, be two Boolean permutations. In this section we give a closed combinatorial formula for the parabolic Kazhdan-Lusztig polynomials of type $q$ indexed by $u$ and $v$. In this formula, there are no restrictions on the subset $J$ of $S$.

Let $(\bar{u}, \bar{v})$ be the left Boolean expressions of $(u, v)$ and consider $\phi_{L}(u, v)$. We start with the following proposition.

Proposition 3.2.1 Suppose that $u, v \leq s_{1} s_{2} \cdots s_{n}$. Then

$$
P_{u, v}^{J, q}(q)= \begin{cases}0, & \text { if } E(u, v)>0 \\ 1, & \text { otherwise }\end{cases}
$$

where $\left.E(u, v)=$| $* \mid 1$ |  |
| :---: | :---: |
| 0 | 1 |
|  | 0 | \right\rvert\, as in Proposition 3.1.1.

Proof. We proceed by induction on $n$, the case $n=1$ being clear. If $v \leq$ $s_{1} s_{2} \cdots s_{n-1}$, then we conclude by induction. So we may assume that $s_{n}$ is the rightmost letter of $\bar{v}$, or, equivalently, we may assume that $v_{n}=1_{l}$. Let us apply Theorem 0.5.14 to $s_{n}$. As $s_{n} \notin v s_{n}$, clearly $\left\{w \leq v s_{n}: w s_{n}<w\right\}=\emptyset$, and hence the sum on the right hand side of the recursive formula of Theorem 0.5.14 is always empty.
If $u_{n}=1_{l}$, then clearly $u s_{n}<u$, and $u \not \leq v s_{n}$ since $s_{n} \leq u$ but $s_{n} \not \leq v s_{n}$. It follows that $\tilde{P}=P_{u s_{n}, v s_{n}}^{J, q}(q)$. So we can conclude by induction.
Suppose that $u_{n}=0$. In this case $u<u s_{n} \notin v s_{n}$ since $s_{n} \leq u s_{n}$ but $s_{n} \notin v s_{n}$. If $u_{n-1}=0$ and $n \in J$, then, by Table $1.2, u s_{n} \notin W^{J}$ and hence $\tilde{P}=0$ as desired. Otherwise, $u s_{n} \in W^{J}$ and $\tilde{P}=P_{u, v s_{n}}^{J, q}(q)$. So the assertion follows by induction.

Note that Proposition 3.2 .1 can be generalized to any Coxeter group $W$.

To simplify notation, we define a map $\gamma:\{(x, y) \in[e,(1, n+1)] \times[e,(1, n+$ 1)]: $x \leq y\} \rightarrow\left\{(x, y) \in\left[e, s_{1} \cdots s_{n}\right] \times\left[e, s_{1} \cdots s_{n}\right]: x \leq y\right\}$ as follows. Given $(x, y)$ in the range, we obtain $\phi_{L}(\gamma(x, y))$ from $\phi_{L}(x, y)$ by the following steps:


\section*{| $*$ |
| :--- |
| O |
|  |}

2. if there are still sub-tableaux of type 000 , go to step (1). Otherwise, change all 2 to $1_{l}$ and all $1_{n}$ to 0 .

For example, suppose that

After the following intermediate steps

$$
\begin{aligned}
& \times \mathrm{O} 0 \times 0 \times \times \times \times \times
\end{aligned}
$$

$$
\begin{aligned}
& \downarrow \\
& \times 00 \times 0 \times \times \times \times \times
\end{aligned}
$$

$$
\begin{aligned}
& \downarrow \\
& \times 00 \times 0 \times \times \times \times \times \\
& \begin{array}{|l|l|l|l|l|l|l|l|l|}
\hline 1_{l} & 1_{l} & 2 & 2 & 2 & 1_{r} & 1_{l} & 2 & 2
\end{array} 1_{l}
\end{aligned}
$$

$$
\begin{aligned}
& \downarrow \\
& \times 0 \circ \times 0 \times \times \times \times \times
\end{aligned}
$$

$$
\begin{aligned}
& \downarrow
\end{aligned}
$$

we obtain
and so $\gamma(x, y)=\left(s_{1} s_{2} s_{4} s_{7} s_{8} s_{10}, s_{1} s_{2} s_{3} s_{4} s_{5} s_{7} s_{8} s_{9} s_{10}\right)$.

Furthermore, we let

$$
a(u, v)=, \quad b(u, v)=\begin{array}{|c|c|}
\hline 2 & 2 \\
\hline * & 0 \\
\hline
\end{array},
$$

and

We drop the $(u, v)$ when no confusion arises.
Then the polynomial $P_{u, v}^{J, q}(q)$ can be computed through the following product formula.

Theorem 3.2.2 The parabolic Kazhdan-Lusztig polynomial $P_{u, v}^{J, q}(q)$ satisfies
where the polynomial $P_{\gamma(u, v)}^{J, q}(q)$ can be computed as in Proposition 3.2.1.
Equivalently,

$$
P_{u, v}^{J, q}(q)= \begin{cases}0, & \text { if } c>0  \tag{3.7}\\ q^{a}(1+q)^{b}, & \text { otherwise }\end{cases}
$$

Proof. In this proof we use both Tits' Word Theorem (Theorem 0.3.6) and Lemma 1.1.2 without explicit mention.

Recall that $(\bar{u}, \bar{v})$ are the left Boolean expressions of $(u, v)$. It is clear that 3.7 is equivalent to 3.6 since $c(u, v)$ is the number of the sub-tableaux nullifying $P_{u, v}^{J, x}(q)$ in 3.6 or nullifying $P_{\gamma(u, v)}^{J, x}(q)$ by Proposition 3.2.1.
Let us prove 3.6 by induction on $l(v)$. If $v \leq s_{1} s_{2} \cdots s_{n}$, we are done because $\gamma(u, v)=(u, v), a(u, v)=0$ and $b(u, v)=0$. So assume $v \not \leq s_{1} s_{2} \cdots s_{n}$. Let $s_{i}$ be the letter at the rightmost place in $\bar{v}$. The recursive property of Theorem 0.5.14 applied to $s_{i}$ gives

$$
\begin{equation*}
P_{u, v}^{J, q}(q)=\tilde{P}-\sum_{w \in\left[u, v s_{i}\right]_{J}: s_{i} \in D_{R}(w)} \mu\left(w, v s_{i}\right) q^{\frac{l(v)-l(w)}{2}} P_{u, w}^{J, q}(q) . \tag{3.8}
\end{equation*}
$$

Let us proceed case by case.

Suppose first that $\phi_{L}(u, v)$ contains one of the following two tableaux:


where, in both cases, $h \in\left\{1_{l}, 0\right\},(f, g) \neq(2,0)$, the column $h$ is the $i$-th, and the column ${ }_{0}$ is the $(i+m)$-th. First of all, by Corollary 0.5 .15 , we can assume $h=0$ as $s_{i} \in D_{R}(v)$ and $\bar{u}\left(s_{i+1}\right)=0$. We claim that, if $m>1$, then

$$
P_{u, v}^{J, q}(q)=q^{|A|}(1+q)^{m-2-|A|} P_{u^{\prime}, v^{\prime}}^{J, q}(q)
$$

where $A=\{k \in[i, i+m-2]: k+1 \in J\}, v^{\prime}=v s_{i} s_{i+1} \cdots s_{i+m-2}$ and $u^{\prime}$ is represented by the expression we obtain by inserting $s_{k}$ to the left in $\bar{u}$ for all $k \in A$. Let us prove the claim. For convenience, we denote by $\overline{v s_{i}}$ the expression we obtain from $\bar{v}$ by deleting the letter $s_{i}$ at the rightmost place. The sum in (3.8) gives no contribution. In fact, let $\bar{w}$ be a reduced expression of an element $w \in\left\{w \leq v s_{i}: w s_{i}<w\right\}$ which is a subword of $\overline{v s_{i}}$. Then $\bar{w}$ has a factor $s_{i}$ on the left and no factors $s_{i+1}$. Hence $s_{i+1} \in D_{R}\left(v s_{i}\right) \backslash D_{R}(w)$ and $\mu\left(w, v s_{i}\right)=0$ by Corollary 0.5 .15 . Let us compute the polynomial $\tilde{P}$. We have

$$
\tilde{P}=q P_{u s_{i}, v s_{i}}^{J, q}(q)+P_{u, v s_{i}}^{J, q}(q)= \begin{cases}q P_{u s_{i}, v s_{i}}^{J, q}(q), & \text { if } i+1 \in J \\ (q+1) P_{u, v s_{i}}^{J, q}(q), & \text { if } i+1 \notin J .\end{cases}
$$

In fact, if $i+1 \in J$, then $s_{i+1} \in D_{R}\left(v s_{i}\right)$ and $u<u s_{i+1} \notin W^{J}$. So, in this case, $P_{u, v s_{i}}^{J, q}(q)=0$. If $i+1 \notin J$, by the induction hypothesis $P_{u s_{i}, v s_{i}}^{J, q}(q)=P_{u, v s_{i}}^{J, q}(q)$. The claim follows by iterating this procedure.
It remains to consider the case $m=1$. Let $\bar{w}$ be a reduced expression of an element $w \in\left\{w \leq v s_{i}: w s_{i}<w\right\}$ which is a subword of $\overline{v s_{i}}$. Then $\bar{w}$ has a factor $s_{i}$ on the left and no factors $s_{i+1}$. In particular, $\bar{w}\left(s_{i}\right)=1$ and $\bar{w}\left(s_{i+1}\right)=0$. Hence, by Corollary 2.4.1, we have that $\mu\left(w, v s_{i}\right)$ can be non-zero only if $l\left(w, v s_{i}\right)=1$ ( $f$ cannot be 0 otherwise $\bar{v}$ would not be reduced). Let us distinguish the three cases: $f=2, f=1_{r}, f=1_{l}$.
If $f=2$, the sum gives no contribution because $l\left(w, v s_{i}\right)>1$ for all possible $w$.

By induction hypothesis,

$$
P_{u, v s_{i}}^{J, q}(q)= \begin{cases}0, & \text { if } i+1 \in J \\ P_{u s_{i}, v s_{i}}^{J, q}(q), & \text { if } i+1 \notin J\end{cases}
$$

and then

$$
\tilde{P}= \begin{cases}q P_{u s_{i}, v s_{i}}^{J, q}(q), & \text { if } i+1 \in J \\ (q+1) P_{u, v s_{i}}^{J, q}(q), & \text { if } i+1 \notin J\end{cases}
$$

as in the case $m>1$.
If $f=1_{r}$ we have

$$
\mu\left(w, v s_{i}\right)= \begin{cases}1, & \text { if } w=v s_{i} s_{i+1} \\ 0, & \text { otherwise }\end{cases}
$$

and the sum contribute exactly with one summand. Hence

$$
P_{u, v}^{J, q}(q)=q P_{u s_{i}, v s_{i}}^{J, q}(q)+P_{u, v s_{i}}^{J, q}(q)-q P_{u, v s_{i} s_{i+1}}^{J, q}(q) .
$$

By induction $P_{u s_{i}, v s_{i}}^{J, q}(q)=P_{u, v s_{i} s_{i+1}}^{J, q}(q)$, thus

$$
P_{u, v}^{J, q}(q)=P_{u, v s_{i}}^{J, q}(q) .
$$

If $f=1_{l}$, we get that $\mu\left(w, v s_{i}\right)$ can be non-zero only if $w$ is the element represented by the expression we obtain from $\overline{v s_{i}}$ by deleting the factor $s_{i+1}$. We have to see if this element $w$ is in $W^{J}$ or not. By Table 1.2, $w$ is not in $W^{J}$ if and only if $i+2 \in J$ and $w_{i+2} \in\left\{2,1_{l}\right\}$. But $w_{i+2}=v_{i+2}$. So
$P_{u, v}^{J, q}(q)= \begin{cases}q P_{u s_{i}, v s_{i}}^{J, q}(q)+P_{u, v s_{i}}^{J, q}(q), & \text { if } i+2 \in J \text { and } v_{i+2}=\left\{2,1_{l}\right\} \\ q P_{u s_{i}, v s_{i}}^{J, q}(q)+P_{u, v s_{i}}^{J,,}(q)-q P_{u, w}^{J, q}(q), & \text { otherwise. }\end{cases}$
By induction hypothesis, $P_{u s_{i}, v s_{i}}^{J, q}(q)=P_{u, v s_{i}}^{J, q}(q)=P_{u, w}^{J, q}(q)$. Hence

$$
P_{u, v}^{J, q}(q)= \begin{cases}(q+1) P_{u, v s_{i}}^{J, q}(q), & \text { if } i+2 \in J \text { and } v_{i+2} \in\left\{2,, 1_{l}\right\} \\ P_{u, v s_{i}}^{J, q}(q), & \text { otherwise } .\end{cases}
$$

We claim that $P_{u, v}^{J, q}(q)=P_{u, v s_{i}}^{J, q}(q)$ in any case, since, if $i+2 \in J$ and $v_{i+2} \in$ $\left\{2,1_{l}\right\}$, then $P_{u, v s_{i}}^{J, q}(q)=0$. In fact, the restrictions on $v_{i+2}$ imply $g \in\left\{2,1_{l}, 0\right\}$, and $i+2 \in J$ forces $g=0$ since $u \in W^{J}$. Hence by induction hypothesis,
$P_{u, v s_{i}}^{J, q}(q)=0$ since columns $i$-th and $(i+1)$-th of $\phi_{L}\left(u, v s_{i}\right)$ form either a

tableau of type 000 or a tableau of type 00.
So the assertion follows by induction.
Now suppose that $\phi_{L}(u, v)$ contains one of the following tableaux:

where the last column is the $(i+1)$-th. Clearly $s_{i} \in D_{R}(v) \backslash D_{R}(u)$. But $u s_{i} \notin W^{J}$, and then $P_{u, v}^{J, q}(q)=0$.

Finally, in all the remaining cases, we have

$$
P_{u, v}^{J, q}(q)=P_{u^{\prime}, v^{\prime}}^{J, q}(q),
$$

where $v^{\prime}=v s_{i}$ and $u^{\prime}$ is the element represented by the expression we obtain from $\bar{u}$ by deleting the letter $s_{i}$ at the rightmostplace, if any. The proof of this fact uses the same technique as above, but is much simpler, and it is left to the reader.
This concludes the induction step and we are done.

Example 8 Let us compute the Kazhdan-Lusztig polynomial $P_{u, v}^{J, x}(q)$ of $S_{10}$, where the Boolean permutations $v$ and $u$, and the subset $J$ of $S$ are as follows:

$$
\begin{aligned}
v & =s_{1} s_{2} s_{3} s_{4} s_{5} s_{7} s_{8} s_{9} s_{8} s_{7} s_{6} s_{5} s_{4} s_{2} s_{1} \\
u & =s_{1} s_{4} s_{9} s_{6} \\
J & =\{2,8\} .
\end{aligned}
$$

By Table 1.2, the permutations $u$ and $v$ are in $S_{10}^{J}$. As the given expressions
are left Boolean, we have

$$
\begin{aligned}
& \times \bigcirc \times \times \times \times \times \bigcirc \times
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& a(u, v)=2 \\
& b(u, v)=1 \\
& c(u, v)=0
\end{aligned}
$$

and using 3.7 we obtain

$$
P_{u, v}^{J, q}(q)=q^{2}(1+q)
$$

## Remarks.

- Theorem 3.2.2, as stated, fails for the right Boolean expressions.
- The result in Theorem 3.2.2 for $J=\emptyset$ (ordinary Kazhdan-Lusztig polynomials) implies Theorem 2.2.1.

We explicitly state the following easy consequence of Theorem 3.2.2. This proves, in the case of Boolean permutations, a conjecture of Brenti ([18]).

Corollary 3.2.3 Let $I \subseteq J$. Then

$$
P_{u, v}^{J, q}(q) \leq P_{u, v}^{I, q}(q)
$$

in the coefficient-wise comparison.
Proof. Straightforward by the analisys of (3.6).

## Part II

## Special matchings and combinatorial invariance

## Chapter 4

## Proof of Lusztig conjecture

This Chapter is devoted to the proof of Lusztig's conjecture on the combinatorial invariance of Kazhdan-Lusztig polynomials for lower Bruhat intervals in any Coxeter group. This follows by proving that special matchings lead to a poset theoretic recursion for computing $R$-polynomials (Corollary 4.4.8). Corollary 4.4.8 is reformulated in a very compact way in Section 5 (Theorem 4.5.2) by introducing a combinatorial version of the Hecke algebra (naturally associated to the special matchings) which acts on the classical Hecke algebra.

### 4.1 Combinatorial properties of Bruhat intervals

In this section we prove some combinatorial properties of Bruhat order on a Coxeter group which are needed in the sequel.

The next result can be proved in a way exactly analogous to Lemma 3.1 of [32], and its proof is therefore omitted. We refer the reader to [39] for a detailed treatment of roots systems.

Lemma 4.1.1 Let $(W, S)$ be a Coxeter system and $t_{1}, \ldots, t_{2 n} \in T \quad(n \in \mathbb{P})$.

- If $t_{1} t_{2}=t_{3} t_{4} \neq e$ then the corresponding positive roots $\alpha_{t_{1}}, \alpha_{t_{2}}, \alpha_{t_{3}}, \alpha_{t_{1} 4}$ are coplanar.
- If $t_{1}, t_{2}, \ldots, t_{n}$ are such that the corresponding positive roots $\alpha_{t_{1}}, \alpha_{t_{2}}, \ldots \alpha_{t_{n}}$ are coplanar then the reflection subgroup $\left\langle t_{1}, t_{2}, \ldots, t_{n}\right\rangle$ is a dihedral reflection subgroup.

Theorem 4.1.2 Let $(W, S)$ be a Coxeter system and $a, b \in W$ be such that either

$$
\begin{equation*}
|\{w \in W: w \triangleleft a, w \triangleleft b\}| \geq 3 \tag{4.1}
\end{equation*}
$$

or

$$
\begin{equation*}
|\{w \in W: w \triangleright a, w \triangleright b\}| \geq 3 . \tag{4.2}
\end{equation*}
$$

Then $a=b$.

Proof. We prove the assertion for (4.1), the proof for (4.2) being entirely similar.

Suppose that $a \neq b$ and let $x, y, z \in\{w \in W: w \triangleleft a, w \triangleleft b\}$. Let $t_{1}, \ldots, t_{6} \in T$ be such that $a t_{1}=x, a t_{3}=y, a t_{5}=z, b t_{2}=x, b t_{4}=y, b t_{6}=z$. Then $a t_{1} t_{2}=$ $a t_{3} t_{4}=a t_{5} t_{6}=b$ so $t_{1} t_{2}=t_{3} t_{4}=t_{5} t_{6} \neq e$. This, by Lemma 4.1.1, implies that $W^{\prime}:=\left\langle t_{1}, \ldots, t_{6}\right\rangle$ is a dihedral reflection subgroup. Clearly, $a, b, x, y, z \in a W^{\prime}$. But, by Lemma 1.4 of [32], $a W^{\prime}$ with the partial order induced by the Bruhat ordering of $W$ is poset-isomorphic to $W^{\prime}$ (considered as an abstract Coxeter system). This is a contradiction since $W^{\prime}$ is a dihedral Coxeter system, and $x, y, z$ are incomparable. Hence $a=b$, as desired.

Note that Theorem 4.1.2 immediately implies Proposition 3.1 of [17]. The following result, though already known, turns out to be a direct consequence of Theorem 4.1.2. We call an interval $[u, v]$ in a poset $P$ dihedral if it is isomorphic to a finite dihedral group ordered by Bruhat order.

Corollary 4.1.3 Let $(W, S)$ be a Coxeter system, and $u, v \in W$. Suppose that $|\{z \in[u, v]: l(z)=l(v)-1\}|=2$. Then $[u, v]$ is a dihedral interval.

Proof. It is well known that, for all $x, y \in W$ such that $y \leq x$ and $l(x)-l(y)=2$, $[y, x]$ is a Boolean algebra of rank 2. Using this and Theorem 4.1.2, it is easy to prove, by induction on $i$, that $|\{w \in[u, v]: l(w)=l(v)-i\}|=2$, for all $i \in[l(v)-l(u)-1]$, as desired.

### 4.2 Pairs of special matchings

The following result follows directly from [17, Lemma 4.1].


Figure 4.1: The orbits $\langle M, N\rangle(u)$ are dihedral intervals

Lemma 4.2.1 Let $P$ be a graded poset, $M$ be a special matching of $P$, and $u, v \in P$ be such that $M(v) \triangleleft v$ and $M(u) \triangleright u$. Then $M$ restricts to a special matching of $[u, v]$.

Since a matching is an application from the set of vertices of a graph to itself, we can compose special matchings as functions. Given two special matchings, $M$ and $N$, we wish to look at the structure of the orbits of $\langle M, N\rangle$, the group generated by $M$ and $N$. For $x \in P$ we denote by $\langle M, N\rangle(x)$ the orbit of $x$ under the action of $\langle M, N\rangle$.

Lemma 4.2.2 Let $P$ be a finite graded poset, and $M$ and $N$ be two special matchings of $P$. Then the orbit $\langle M, N\rangle(u)$ of any $u \in P$ is a dihedral interval.

Proof. Since $P$ is finite, the orbit $\langle M, N\rangle(u)$ is also finite. Therefore there exists $x \in\langle M, N\rangle(u)$ such that $M(x) \triangleleft x$ and $N(x) \triangleleft x$. If $M(x)=N(x)$ then $\langle M, N\rangle(u)=\{x, M(x)\}$ and we are done. Else, by the definition of a special matching we have that $N(M(x)) \triangleleft M(x), N(M(x)) \triangleleft N(x), M(N(x)) \triangleleft$ $N(x)$, and $M(N(x)) \triangleleft M(x)$. If $M(N(x))=N(M(x))$ then $\langle M, N\rangle(u)=$ $\{x, N(x), M(x), N(M(x))\}$ and we are done. Otherwise we conclude, similarly, that $M N M(x) \triangleleft N M(x), M N M(x) \triangleleft M N(x), N M N(x) \triangleleft M N(x)$, and $N M N(x) \triangleleft N M(x)$ (see Figure 4.1).

If $M N M(x)=N M N(x)$ then we are done, else we continue in this way. Since $\langle M, N\rangle(u)$ is finite there exists $l \in \mathbb{P}$ such that $\underbrace{M N M \ldots}_{l}(x)=\underbrace{N M N \ldots}_{l}(x)$ and the result follows.

The following is the main result of this section, and one of the key ingredients in the proof of our main result. We say that a graded poset $P$ avoids $K_{3,2}$ if there are no elements $a_{1}, a_{2}, a_{3}, b_{1}, b_{2} \in P$, all distinct, such that either $a_{i} \triangleleft b_{j}$
for all $i \in[3], j \in[2]$ or $a_{i} \triangleright b_{j}$ for all $i \in[3], j \in[2]$. So, for example, a Coxeter group under Bruhat order avoids $K_{3,2}$ by Theorem 4.1.2.

Proposition 4.2.3 Let $P$ be a graded poset that avoids $K_{3,2}, v \in P$, and $M$ and $N$ be two special matchings of $P$ such that $M(v) \neq N(v)$. Let $v^{\prime} \notin\{M(v), N(v)\}$ and suppose that either
i) $M(v) \triangleleft v, N(v) \triangleleft v$ and $v^{\prime} \triangleleft v$, or
ii) $M(v) \triangleright v, N(v) \triangleright v$ and $v^{\prime} \triangleright v$.

Then

$$
|\langle M, N\rangle(v)|=\left|\langle M, N\rangle\left(v^{\prime}\right)\right| .
$$

Proof. We prove the statement only in case i), case ii) following by considering the dual poset $P^{*}$. Suppose that $|\langle M, N\rangle(v)|=2 n,\left|\langle M, N\rangle\left(v^{\prime}\right)\right|=2 m$. Note that, since $v^{\prime} \notin\{M(v), N(v)\},\langle M, N\rangle(v) \cap\langle M, N\rangle\left(v^{\prime}\right)=\emptyset$. Therefore, no element of $\langle M, N\rangle(v)$ is matched by either $M$ or $N$ to an element of $\langle M, N\rangle\left(v^{\prime}\right)$. This, by the definition of a special matching, and a simple induction on $k$, implies that

$$
\underbrace{M N M \cdots}_{k}\left(v^{\prime}\right) \triangleleft \underbrace{M N M \cdots}_{k}(v) \quad, \quad \underbrace{M N M \cdots}_{k}\left(v^{\prime}\right) \triangleleft \underbrace{N M N \cdots}_{k-1}\left(v^{\prime}\right),
$$

and similarly that

$$
\underbrace{N M N \cdots}_{k}\left(v^{\prime}\right) \triangleleft \underbrace{N M N \cdots}_{k}(v) \quad, \quad \underbrace{N M N \cdots}_{k}\left(v^{\prime}\right) \triangleleft \underbrace{M N M \cdots}_{k-1}\left(v^{\prime}\right),
$$

for all $k \in[n]$. Therefore, $m \geq n$. If $m>n$, then $\underbrace{M N M \cdots}_{n}(v)=\underbrace{N M N \cdots}_{n}(v)$, while $\underbrace{M N M \cdots}_{n}\left(v^{\prime}\right) \neq \underbrace{N M N \cdots}_{n}\left(v^{\prime}\right)$, and this contradicts the fact that $P$ avoids $K_{3,2}$ (see Figure 4.2).

We now restrict our attention to the case where $P$ is a lower Bruhat interval of a Coxeter group $W$, i.e. an interval of the form $[e, v]$, with $v \in W$. In this case we often refer to a special matching of $[e, v]$ as a special matching of $v$.

Lemma 4.2.4 Let $u, v \in W, u \leq v$ and $M$ and $N$ be two special matchings of $v$. Suppose that $|\langle M, N\rangle(u)|>2$. Then there exists a unique maximal dihedral interval I containing $\langle M, N\rangle(u)$. Furthermore I is a union of orbits of $\langle M, N\rangle$.


Figure 4.2: The case $n=3$ and $m>n$

Proof. The result follows easily by Theorem 4.1.2, Lemma 4.2.2 and the definition of a special matching.

Lemma 4.2.5 Let $u, v \in W, u \leq v$ and $M$ and $N$ be two special matchings of v. If $|\langle M, N\rangle(u)|=2 m>2$, then there exists $u^{\prime}$ and a dihedral interval I such that $e, M(e), N(e) \in I,\left|\langle M, N\rangle\left(u^{\prime}\right)\right|=2 m$ and $\langle M, N\rangle\left(u^{\prime}\right) \subseteq I$. In particular, if $M(e) \neq N(e)$, then $W_{\{M(e), N(e)\}}$ contains an orbit of cardinality $2 m$.

Proof. Without loss of generality we may assume that $M(u), N(u) \triangleleft u$. We claim that we can find a sequence $u=u_{1} \triangleright u_{2} \triangleright \cdots \triangleright u_{k}$ such that $M\left(u_{i}\right), N\left(u_{i}\right) \triangleleft$ $u_{i},\left|\langle M, N\rangle\left(u_{i}\right)\right|=2 m$ for all $i \in[k]$, and $\left[e, u_{k}\right]$ is a dihedral interval. In fact if $\{z \in[e, u]: z \triangleleft u\}=\{M(u), N(u)\}$ then we are done. Otherwise let $u_{2} \in\{z \in$ $[e, u]: z \triangleleft u\} \backslash\{M(u), N(u)\}$. Then, by Proposition 4.2.3, $\left|\langle M, N\rangle\left(u_{2}\right)\right|=2 m$ and $M\left(u_{2}\right) \triangleleft u_{2}, N\left(u_{2}\right) \triangleleft u_{2}$. If $\left\{z \in\left[e, u_{2}\right]: z \triangleleft u_{2}\right\}=\left\{M\left(u_{2}\right), N\left(u_{2}\right)\right\}$ then our claim is proved. Otherwise let $u_{3} \in\left\{z \in\left[e, u_{2}\right]: z \triangleleft u_{2}\right\} \backslash\left\{M\left(u_{2}\right), N\left(u_{2}\right)\right\}$ and continue as above. This proves our claim. Let $I$ be the maximal dihedral interval containing $\langle M, N\rangle\left(u_{k}\right)$. Since $\left[e, u_{k}\right]$ is dihedral we have $\langle M, N\rangle\left(u_{k}\right) \subseteq$ $\left[e, u_{k}\right] \subseteq I$ and by Lemma 4.2.4 $I$ is union of orbits of $\langle M, N\rangle$. In particular $M(e), N(e) \in I$ and the proof is complete.

### 4.3 Groups of rank 3

If $J \subseteq S$ and $w \in W$ we let $W_{J}(w):=W_{J} \cap[e, w]$.
For $x, y \in S$ we denote by $\cdots x y x$ (respectively $x y x \cdots$ ) a word given by alternating $x$ and $y$ that ends (respectively begins) with $x$. Inside any single proof, if the length of such a word is not specified, it is assumed to be arbitrary but fixed.

A complete matching of an interval $[e, w]$ is called a multiplication matching if there exists $s \in S$ such that either $M=\lambda_{s}$ or $M=\rho_{s}$.

The expressions considered for an element of a Coxeter group are always assumed to be reduced.

Lemma 4.3.1 Let $u, w \in W, u \leq w$ and $M$ be a special matching of $w$. Suppose that $u$ does not belong to any dihedral interval containing $e$ and $M(e)$, and that $M(u) \triangleright u$. Then there exist two distinct elements $u_{1}$ and $u_{2}$ such that $u_{i} \triangleleft u$ and $M\left(u_{i}\right) \triangleright u_{i}$, for $i=1,2$.

Proof. By Lemma 4.2.1, given an element $v$ with $v \triangleright M(v), M$ restricts to a special matching of $[e, v]$. In particular $M(e) \leq v$. Hence, if $M(e) \notin u$, then $M(x) \triangleright x$ for all $x \in[e, u]$, and the assertion is proved.
So we may assume that $M(e) \leq u$. Hence the interval $[e, u]$ is not dihedral and, in particular, $[e, M(u)]$ has at least two coatoms distinct from $u$, say $x_{1}$ and $x_{2}$. Then the elements $u_{i}=M\left(x_{i}\right)$, for $i=1,2$, satisfy the conditions of the statement.

Lemma 4.3.2 Let $u, w \in W, u \leq w$ and $M$ be a special matching of $w$. Suppose that for all $x \leq u$ such that $x$ belongs to a dihedral interval containing e and $M(e)$ we have $M(x)=x M(e)$. Then $M(u)=u M(e)$.

Proof. We proceed by induction on $l(u)$ the statement being trivial if $l(u)=$ 0 . We may assume $M(u) \triangleright u$, otherwise the statement follows by induction. Furthermore, we may clearly assume that $u$ does not belong to a dihedral interval containing $e$ and $M(e)$. Hence, by Lemma 4.3.1, there exist two distinct elements $u_{1}$ and $u_{2}$ such that $u_{i} \triangleleft u$ and $M\left(u_{i}\right) \triangleright u_{i}$, for $i=1,2$. By our induction hypothesis $M\left(u_{i}\right)=u_{i} M(e)$, for $i=1,2$. Therefore $u M(e)$ covers $u, M\left(u_{1}\right)$ and $M\left(u_{2}\right)$ and, by the definition of a special matching, $M(u)$ also covers $u, M\left(u_{1}\right)$ and $M\left(u_{2}\right)$. Hence $M(u)=u M(e)$ by Theorem 4.1.2.

Proposition 4.3.3 Let $w \in W$ and $M$ be a special matching of $w$. Then for all $J \subseteq S$ such that $M(e) \in J, M$ stabilizes $W_{J}(w)$.


Figure 4.3: Proof of Lemma 4.3.4.

Proof. We prove that $u \in W_{J}(w)$ implies $M(u) \in W_{J}(w)$ by induction on $l(u)$, this being trivial if $l(u)=0$. We may clearly assume that $M(u) \triangleright u$. Let $x \triangleleft M(u), x \neq u$. Then $M(x) \triangleleft u$ and by our induction hypothesis $x \in W_{J}(w)$. Hence all the coatoms of $M(u)$ are in $W_{J}(w)$, so $M(u) \in W_{J}(w)$.

From now on we assume that $(W, S)$ is a Coxeter system of rank 3. We let $S:=\{s, r, t\}, w \in W, M$ be a special matching of $w$ and we assume that $M(e)=s$.

Lemma 4.3.4 If $r s, s r, t s, s t \leq w, r s \neq s r$, $s t \neq t s, M(t)=t s$ and $M(r)=r s$, then $M(s t)=$ sts and $M(s r)=s r s$.

Proof. By symmetry it suffices to show that $M(s t)=s t s$.
By definition of a special matching $M(s t) \triangleright s t$ and $M(s t) \triangleright t s$, so $M(s t) \in$ $\{s t s, t s t\}$. Similarly, $M(s r) \in\{s r s, r s r\}$. Suppose $M(s t)=t s t$. If $s t r \leq w$ then (see Figure 4.3) $M(s t r) \triangleright t s t, M(s r)$. But there are no elements covering both tst and $M(s r)$, so str $\not \leq w$. Similarly srt $\not \leq w$. Now consider a reduced expression for $w$. Then $t s t$ and either srs or $r s r$ are both subexpressions of it and it is easy to see that these conditions force that either str or srt is also a subexpression, contradicting the fact that str $\not \leq w$ and srt $\not \approx w$.

Lemma 4.3.5 Suppose $r s, s r, t s$, st $\leq w, M(t)=t s$ and $M(r)=r s$, but $M \neq$ $\rho_{s}$. Let $x_{0}$ be a minimal element such that $M\left(x_{0}\right) \neq x_{0} s$. By Lemma 4.3.2, we necessarily have $x_{0} \in W_{\{s, t\}}(w) \cup W_{\{s, r\}}(w)$ and we assume that $x_{0} \in W_{\{s, t\}}(w)$. Let $u$ be such that $x_{0} \triangleleft u \leq w$ and $u \notin W_{\{s, t\}}(w)$. Then $u \in\left\{x_{0} r, r x_{0}\right\}$. Furthermore, if $s r \neq r s$, then $u=r x_{0}$.

Proof. Clearly, $s \notin D_{R}\left(x_{0}\right)$, and $M\left(x_{0}\right) \triangleright x_{0}$. Let $x_{0}=\underbrace{\alpha \beta \alpha \cdots t s t}_{k}$ where $\alpha=s$ if $k$ is even, $\alpha=t$ if $k$ is odd and $\{\alpha, \beta\}=\{s, t\}$. Since $x_{0} \neq t$ we conclude that $s t \neq t s$. Hence, by Lemma 4.3.4, $M(s r)=s r s$. Let $u$ be as in the statement and assume $u \notin\left\{x_{0} r, r x_{0}\right\}$ if $s r=r s$ and $u \neq r x_{0}$ if $s r \neq r s$. So $u$ is obtained by inserting a letter $r$ in the unique reduced expression of $x_{0}$.

Let $y:=\alpha u$. Then $y \triangleleft u$, hence the elements in $W_{\{s, t\}}(y)$ are all strictly smaller than $x_{0}$. Furthermore, the elements in $W_{\{s, r\}}(y)$ are smaller than, or equal to, srs. Hence, by Lemma 4.3.2, $M(y)=y s$. Since $x_{0}$ and $y$ are both covered by $u, M(u) \triangleright M\left(x_{0}\right)=\underbrace{\beta \alpha \beta \cdots t s t}_{k+1} \neq \underbrace{\alpha \beta \alpha \cdots s t s}_{k+1}$ and $M(u) \triangleright M(y)$. Then it is not difficult to see that these two conditions force $M(u)=y s t$ which is a contradiction since, as one can verify, yst $\ngtr u$.

Lemma 4.3.6 Suppose that $M(t)=t s \neq s t$ and $M(r)=s r \neq r s$. Then rst $\not \leq w$. Furthermore, if $r t \neq t r$, then $r t \not \leq w$.

Proof. Suppose $r t \leq w$. Then, by the definition of special matching, $M(r t) \triangleright r t$, $M(r t) \triangleright t s$ and $M(r t) \triangleright s r$ (see Figure 4.4). If $r t \neq t r$ there are no such elements and this proves the second part of the statement. If $r t=t r$ then necessarily $M(r t)=t s r$. If $r s t \leq w$ then $M(r s t)$ would cover both $t s r$ and $r s t$ and there are clearly no such elements.

In the following results we distinguish three cases:

1. $M(t)=t s, M(r)=r s \neq r s$ and $M \not \equiv \rho_{s}$. We let $x_{0}$ be a minimal element such that $M\left(x_{0}\right) \neq x_{0} s$, we assume that $x_{0} \in W_{\{s, t\}}(w)$ and we let $\alpha \beta \alpha \cdots$ tst be its unique reduced expression.
2. $M(t)=t s, M(r)=r s=s r$ and $M \not \equiv \rho_{s}$. We let $x_{0}$ be the minimal element such that $M\left(x_{0}\right) \neq x_{0} s$ and we let $\alpha \beta \alpha \cdots$ tst be its unique reduced expression.
3. $M(t)=t s \neq s t$ and $M(r)=s r \neq r s$.


Figure 4.4: Proof of Lemma 4.3.6.

The following Proposition shows that if an interval $[e, w]$ has a special matching which is not a multiplication matching, then $w$ must be of a special form.

Proposition 4.3.7 In case (1) any element $u \leq w$ has a reduced expression $u=\cdots r \beta r \eta \alpha \beta \alpha \cdots$, where $\eta \in\{e, \beta\}$;

In case (2) any element $u \leq w$ has a reduced expression of the form $u=$ $\cdots r \beta r \eta(\alpha \beta \alpha \cdots) \delta$, where $\eta \in\{e, \beta\}$ and $\delta \in\{e, r\}$;

In case (3) any element $u \leq w$ has a reduced expression $u=\cdots$ tstहrsr $\cdots$, where $\varepsilon \in\{e, s\}$.

Proof. It is clear that in all cases it is enough to prove the statement for $u=w$, the general result following by the subword property.
(1) Let $\alpha \beta \alpha \cdots$ tst be a longest subword of a reduced expression of $w$ given by alternating $s$ and $t$, starting with $\alpha$ and ending with $t$ with the first $\alpha$ chosen as left as possible. Consider the first letter $r$ that appears after the first $\alpha$ of this subword. By Lemma 4.3.5, this letter $r$ can be pushed to the left of this subword. Hence we obtain a reduced expression for $w$ where no $r$ appears after the first letter $\alpha$ and the thesis follows.
(2) This is similar to the proof of (1) but in this case a letter $r$ can also appear on the right of the longest subword of the form $\alpha \beta \alpha \cdots t s t$ and we are done.
(3) Consider a reduced expression for $w$ and look at the rightmost letter $t$ and at the leftmost letter $r$ of this reduced expression. If this $t$ appears on the left of this $r$ we are done. Otherwise, by Lemma 4.3.6, there cannot be a letter $s$ between them and $r t=t r$. So these two letters are adjacent and hence we can find a reduced expression for $w$ in which all the letters $t$ appear before all the letters $r$ and the result follows.

Proposition 4.3.8 There exists $x \in\{r, t\}$ such that either $M \equiv \lambda_{s}$ or $M \equiv \rho_{s}$ on $W_{\{s, x\}}(w)$.

Proof. We may assume that $[e, w]$ is not dihedral. Note that the result is true for a special matching $M$ of $[e, w]$ if and only if it is true for the special matching $\tilde{M}$ of $\left[e, w^{-1}\right]$ defined by $\tilde{M}(x):=\left(M\left(x^{-1}\right)\right)^{-1}$, for all $x \leq w^{-1}$. We may clearly assume that $M$ is not a multiplication matching and that

$$
\begin{equation*}
4 \notin\left\{\left|W_{\{r, s\}}(w)\right|,\left|W_{\{t, s\}}(w)\right|\right\} \tag{4.3}
\end{equation*}
$$

In particular, $r s \neq s r$ and $t s \neq s t$ If $M(r)=r s$ and $M(t)=t s$ we are in case (1) (possibly by exchanging the roles of $r$ and $t$ ). If $M(r)=s r$ and $M(t)=s t$ then $\tilde{M}$ is in case (1). If $M(r)=s r \neq r s$ and $M(t)=t s \neq s t$ we are in case (3). So we only need to consider these two cases.

In case (1) we have that $\beta=s$ otherwise, by Proposition 4.3.7, $W_{\{r, s\}}(w)=$ $\{e, s, r, r s\}$ and this is not possible by (4.3). By contradiction, suppose that $M \not \equiv \rho_{s}$ on $W_{\{r, s\}}(w)$, and let $y_{0} \in W_{\{r, s\}}(w)$ be a minimal element such that $M\left(y_{0}\right) \neq y_{0} s$. By Lemma 4.3.5, $y_{0} t \nless w$, but this is a contradiction, since $w$ is not dihedral.
In case (3) we claim that either $M \equiv \rho_{s}$ on $W_{\{t, s\}}(w)$ or $M \equiv \lambda_{s}$ on $W_{\{r, s\}}(w)$. We prove this statement by induction on $l(w)$. By Proposition 4.3.7 $w=$ $\underbrace{\cdots t s t}_{k} \varepsilon \underbrace{r s r \cdots}_{h}$ (this being a reduced expression), where $\varepsilon \in\{e, s\}$. By (4.3) we have $h, k \geq 2$. Let $w_{1}$ and $w_{2}$ be the two coatoms of $[e, w]$ obtained by deleting, respectively, the first and the last letter of this reduced expression of $w$. Clearly, there exists $i \in\{1,2\}$ such that $M$ restricts to a special matching of $\left[e, w_{i}\right]$. We assume $i=1$ the case $i=2$ being similar. By our induction hypothesis either $M \equiv \rho_{s}$ on $W_{\{t, s\}}\left(w_{1}\right)$ or $M \equiv \lambda_{s}$ on $W_{\{r, s\}}\left(w_{1}\right)$. In this second case we are done since $W_{\{r, s\}}\left(w_{1}\right)=W_{\{r, s\}}(w)$. So assume that $M \equiv \rho_{s}$ on $W_{\{t, s\}}\left(w_{1}\right)$. But $W_{\{t, s\}}(w) \backslash W_{\{t, s\}}\left(w_{1}\right)=\{\underbrace{\cdots t s t}_{k}, \underbrace{\cdots s t s}_{k+1}\}$ and since, by Propo-
sition 4.3.3, $M$ stabilizes $W_{\{t, s\}}(w)$ we necessarily have $M(\underbrace{\cdots t s t}_{k})=(\underbrace{\cdots s t s}_{k+1})$ and hence $M \equiv \rho_{s}$ on $W_{\{t, s\}}(w)$.

Proposition 4.3.8 allows us to add some hypothesis to the cases we are dealing with, without affecting the generality of our argument.
(1') $M(t)=t s, s r \neq r s, M \equiv \rho_{s}$ on $W_{\{s, r\}}(w)$ and $M \not \equiv \rho_{s}$ on $W_{\{s, t\}}(w)$. We let $x_{0}$ be the minimal element such that $M\left(x_{0}\right) \neq x_{0} s$ and we let $\alpha \beta \alpha \cdots t s t$ be its unique reduced expression.
(2') $M(t)=t s, r s=s r$ and $M \not \equiv \rho_{s}$ on $W_{\{s, t\}}(w)$. We let $x_{0}$ be the minimal element such that $M\left(x_{0}\right) \neq x_{0} s$ and we let $\alpha \beta \alpha \cdots$ tst be its unique reduced expression.
(3') $M(t)=t s \neq s t, s r \neq r s$ and $M \equiv \lambda_{s}$ on $W_{\{s, r\}}(w)$.
The next result describes how a special matching acts on the interval $[e, w]$.
Proposition 4.3.9 In case ( $1^{\prime}$ ) let $u \leq w, u=\cdots r \beta r \eta \alpha \beta \alpha \cdots$ where $\eta \in$ $\{e, \beta\}$ and $\beta \notin D_{R}(\cdots r \beta r)$. Then $M(u)=\cdots r \beta r M(\eta \alpha \beta \alpha \cdots)$.

In case (2') let $u \leq w, u=\cdots r \beta r \eta(\alpha \beta \alpha \cdots) \delta$ where $\eta \in\{e, \beta\}, \delta \in\{e, r\}$ and $\beta \notin D_{R}(\cdots r \beta r)$. Then $M(u)=\cdots r \beta r M(\eta \alpha \beta \alpha \cdots) \delta$.

In case (3') let $u \leq w, u=\cdots$ tstहrsr $\cdots$ where $\varepsilon \in\{e, s\}$ and $s \notin$ $D_{L}(r s r \cdots)$. Then $M(u)=M(\cdots t s t \varepsilon) r s r \cdots$.

Proof. (1') We proceed by induction on $l(u)$ the case $\cdots r \beta r=e$ being trivial and the case $\eta \beta \alpha \beta \cdots=e$ following by Lemma 4.3.2 if $\beta=t$ and by our hypotheses if $\beta=s$.

So suppose that the length of the string $\cdots r \beta r$ is at least 1 . We may assume that $M(\eta \alpha \beta \alpha \cdots) \triangleright \eta \alpha \beta \alpha \cdots \neq e$, else the statement follows by our induction hypothesis. Now let $x \in D_{L}(\cdots r \beta r)$. Then $x u \triangleleft u$ and by our induction hypothesis $M(x u)=x(\cdots r \beta r) M(\eta \alpha \beta \alpha \cdots)$. Now let $v$ be the unique element such that $v \triangleleft \eta \alpha \beta \alpha \cdots$ and $M(v) \triangleright v$. Then $\cdots r \beta r v \triangleleft u$ and $M(\cdots r \beta r v)=\cdots r \beta r M(v)$ by our induction hypothesis. Since $\cdots r \beta r M(\eta \alpha \beta \alpha \cdots)$ covers $u, M(x u)$ and $M(\cdots r \beta r v)$ and these three elements are clearly distinct, we necessarily have $M(u)=\cdots r \beta r M(\eta \alpha \beta \alpha \cdots)$.
(2') We proceed by induction on $l(u)$. We may assume that $M(\eta \alpha \beta \alpha \cdots) \triangleright$ $\eta \alpha \beta \alpha \cdots$ as otherwise the statement follows by our induction hypothesis. Suppose first that $\cdots r \beta r=e$. Then we can assume $\delta=r$ and $\eta \alpha \beta \alpha \cdots \neq e$ as otherwise the result would be trivial. So, if we define $v$ as in case (1'),
we have that $v r$ and $\eta \beta \alpha \beta \cdots$ are both covered by $u$. Then $M(u)$ is necessarily equal to $M(\eta \alpha \beta \alpha \cdots) r$ since this is the unique element that covers $u$, $M(v r)=M(v) r$ and $M(\eta \beta \alpha \alpha \cdots)$ and the result follows similarly. If $\cdots r \beta r \neq e$ and $\eta \beta \alpha \beta \cdots=e$ the claim follows from Lemma 4.3 .2 and if $\cdots r \beta r \neq e$ and $\eta \beta \alpha \beta \cdots \neq e$ the proof is similar to the case ( $1^{\prime}$ ).

Case (3') is very similar to case ( $1^{\prime}$ ) and is left to the reader.
The next result gives some further restrictions on a special matching which is not a multiplication matching.

Proposition 4.3.10 In case ( $1^{\prime}$ ) let $w=\underbrace{\cdots r \beta r}_{h} \eta \alpha \beta \alpha \cdots$, with $\eta \in\{e, \beta\}$ and $\beta \notin D_{R}(\cdots r \beta r)$. If $h \geq 2$ and $\beta \in D_{L}(w)$, then $M \circ \lambda_{\beta}=\lambda_{\beta} \circ M$.

In case (2') let $w=\underbrace{\cdots r \beta r}_{h} \eta(\alpha \beta \alpha \cdots) \delta$, with $\eta \in\{e, \beta\}, \delta \in\{e, r\}$ and
$\beta \notin D_{R}(\cdots r \beta r)$. If $h \geq 2$ and $\beta \in D_{L}(w)$, then $M \circ \lambda_{\beta}=\lambda_{\beta} \circ M$.
In case (3') let $w=\cdots$ tstを $\underbrace{r s r \cdots}_{h}$, with $\varepsilon \in\{e, s\}$ and $s \notin D_{L}(r s r \cdots)$. If $h \geq 2$ and $s \in D_{R}(w)$, then $M \circ \rho_{s}=\rho_{s} \circ M$.

Proof. By Lemma 4.2.5, we know that two special matchings $M$ and $N$ of a Bruhat interval $[e, w]$ commute if and only if they do inside the dihedral intervals containing $M(e)$ and $N(e)$.
In cases ( $1^{\prime}$ ) and ( $2^{\prime}$ ), since $M \equiv \rho_{s}$ on $W_{\{r, s\}}(w)$ it is clear that $M \circ \lambda_{\beta}=\lambda_{\beta} \circ M$ on $W_{\{r, s\}}(w)$. So we only have to show that $M \circ \lambda_{\beta}=\lambda_{\beta} \circ M$ on $W_{\{t, s\}}(w)$.

Let $u:=\underbrace{\beta \alpha \beta \cdots}_{k} \leq w$. We claim that if $M(u) \triangleright u$ then $M(u)=\underbrace{\beta \alpha \beta \cdots}_{k+1}$. In fact, consider $v:=\beta r \underbrace{\alpha \beta \alpha \cdots}_{k-1}$. It is clear that $u \triangleleft v \leq w$. By Proposition 4.3 .9 we have that $M(v)=\beta r M(\underbrace{\alpha \beta \alpha \cdots}_{k-1})$. Since, by the definition of a special matching, $M(v) \triangleright M(u)$ we necessarily have $M(\underbrace{\alpha \beta \alpha \cdots}_{k-1}) \triangleright \underbrace{\alpha \beta \alpha \cdots}_{k-1}$. By Proposition 4.3.3, $M(\underbrace{\alpha \beta \alpha \cdots}_{k-1}), M(u) \in W_{\{s, t\}}(w)$, so $M(v)=\beta r \underbrace{\alpha \beta \alpha \cdots}_{k}$ and $M(u)=\underbrace{\beta \alpha \beta \cdots}_{k+1}$.

Now consider an orbit of $\left\langle M, \lambda_{\beta}\right\rangle$ inside $W_{\{s, t\}}(w)$ of cardinality greater than 2 . We show that the cardinality of this orbit is necessarily 4 . Let $z$ be the smallest element of this orbit, say $z=\underbrace{\alpha \beta \alpha \cdots}_{k-1}$. Then $\lambda_{\beta}(z)=\underbrace{\beta \alpha \beta \cdots}_{k}$, forcing $M(z)=\underbrace{\alpha \beta \alpha \cdots}_{k}$. Then by our claim $M\left(\lambda_{\beta}(z)\right)=\underbrace{\beta \alpha \beta \cdots}_{k+1}=\lambda_{\beta}(M(z))$.

The proof of case (3') is very similar and is left to the reader.

The following result is not needed in the sequel and is a particular case of Theorem 4.4.7. Nevertheless we state and prove it to complete the discussion on groups of rank 3 .

Theorem 4.3.11 Let $(W, S)$ be a Coxeter system of rank 3, w $\in W, l(w)>1$ and $M$ be a special matching of $[e, w]$. Suppose that $[e, w]$ is not a dihedral interval. Then there exists a multiplication matching $N$ of $[e, w]$ such that

1. $N(M(u))=M(N(u))$, for all $u \leq w$;
2. $N(w) \neq M(w)$.

Proof. We can clearly assume that $M$ is not a multiplication matching. In fact, if $M=\lambda_{s}$ then $w$ has a reduced expression having $s$ as a first letter $w=s s_{1} \cdots s_{k}$ and hence it is enough to set $N=\rho_{s_{k}}$, and similarly if $M=\rho_{s}$. Note also that the statement is true for $M$ if and only if it is for the special matching $\tilde{M}$ defined in the proof of Proposition 4.3.8. If there exists a $t \in S$ such that $M$ is not a multiplication matching on $W_{\{s, t\}}(w)$ then, by Proposition 4.3.8, either $M$ or $\tilde{M}$ is in one of the cases ( $\left.1^{\prime}\right),\left(2^{\prime}\right)$ or ( $\left.3^{\prime}\right)$. If such a $t$ does not exist we are necessarily in case ( $3^{\prime}$ ). So we are reduced to consider these 3 cases. In cases (1') and (2'), if $r \in D_{L}(w)$ it is enough to take $N=\lambda_{r}$. Otherwise we necessarily have $\beta \in D_{L}(w)$ and $\beta r \neq r \beta$. Then, by Proposition 4.3.10, $M \circ \lambda_{\beta}=\lambda_{\beta} \circ M$ and, by Proposition 4.3.9, $M(w) \neq \lambda_{\beta}(w)$. Once again case $\left(3^{\prime}\right)$ is similar and is left to the reader.

### 4.4 Main result

Now we face the problem of a special matching of an interval $[e, w]$ where $w$ belongs to an arbitrary Coxeter group. We recall the following result for future references. It follows by the proof of Theorem 5.2 of [17] and in fact holds in much more generality (see Theorem 7.2.3).

Theorem 4.4.1 Let $(W, S)$ be a Coxeter system, $w \in W$ and $M$ be a special matching of $[e, w]$. Suppose that, for all $v \leq w$ with $M(v) \triangleleft v$, there exists a multiplication matching $N_{v}$ of $[e, v]$ such that $M N_{v} \equiv N_{v} M$ and $M(v) \neq N_{v}(v)$. Then

$$
\widetilde{R}_{u, w}(q)=\widetilde{R}_{M(u), M(w)}(q)+\chi(M(u) \triangleright u) q \widetilde{R}_{u, M(w)}(q)
$$

for all $u \leq w$.


Figure 4.5: Proof of Lemma 4.4.3.

Lemma 4.4.2 Let $w \in W$ and $M$ be a special matching of $[e, w]$ with $M(e)=s$. Then there exists at most one $x \in S$ such that $M \not \equiv \lambda_{s}$ and $M \not \equiv \rho_{s}$ on $W_{\{s, x\}}(w)$.

Proof. Suppose there are 2 such elements, say $t$ and $r$. It is known that for all $J \subseteq S$ there exists a unique maximal element in $W_{J}(w)$ that we denote $w[J]$, so that $W_{J}(w)=[e, w[J]]$. By Proposition 4.3.3, $M$ restricts to a special matching of $[e, w[\{s, r, t\}]]$. But this contradicts with Proposition 4.3.8.

Lemma 4.4.3 Let $w \in W, M$ be a special matching of $[e, w]$ and $s=M(e)$. Let $t, r \in S$ be such that $M(t)=t s \neq s t$ and $M(r)=s r \neq r s$ and let $k_{1}, \ldots, k_{p} \in S \backslash$ $\{s\}, p \in \mathbf{N}$, be such that $k_{j} s=s k_{j}$ for $j \in[p]$. Suppose that $r k_{1} \cdots k_{p} t \leq w$ and $l\left(r k_{1} \cdots k_{p} t\right)=p+2$. Then there exist $h_{1}, \ldots h_{p} \in S$ such that $\left\{k_{1}, \ldots, k_{p}\right\}=$ $\left\{h_{1}, \ldots, h_{p}\right\}$ and $i \in[0, p]$ such that $r k_{1} \cdots k_{p} t=h_{1} \cdots h_{i} \operatorname{tr} h_{i+1} \cdots h_{p}$.

Proof. By Proposition 4.3.3 and Lemma 4.3 .6 (applied to the interval $[e, w[J]]$, where $J:=\{s, r, t\}$ ), we have that $t r=r t$, so the result holds if $p=0$. We proceed by induction on $p$. Let $u:=r k_{1} \cdots k_{p} t$. It suffices to show that either $D_{L}(u) \neq\{r\}$ or $D_{R}(u) \neq\{t\}$, the result then following by induction on $p$. It is clear that $k_{1} \cdots k_{p} t \triangleleft u$. Furthermore, by Lemma 4.3.2, $M\left(k_{1} \cdots k_{p} t\right)=$ $k_{1} \cdots k_{p} t s$ and similarly $M\left(r k_{1} \cdots k_{p}\right)=s r k_{1} \cdots k_{p}$ (see Figure 4.5). Therefore, since $M$ is a special matching, $M(u) \triangleright u, M(u) \triangleright k_{1} \ldots k_{p} t s$, and $M(u) \triangleright$ $s r k_{1} \ldots k_{p}$. If $r$ is the unique left descent of $u$ and $t$ is its unique right descent then necessarily either $r \in D_{L}(M(u))$ or $t \in D_{R}(M(u))$ (or both). Suppose $r \in D_{L}(M(u))$ the other case being similar. Since $r \not \leq k_{1} \cdots k_{p} t s$ and $M(u) \triangleright$
$k_{1} \cdots k_{p} t s$ we have $M(u)=r k_{1} \cdots k_{p} t s$. Now, since $r k_{1} \cdots k_{p} t s \triangleright s r k_{1} \cdots k_{p}$ and $t \not \leq s r k_{1} \cdots k_{p}$ we have $r k_{1} \cdots k_{p} s=s r k_{1} \cdots k_{p}$, which implies $s r=r s$ and this is a contradiction.

Proposition 4.4.4 Let $J:=\{r \in S: M(r)=s r\}$ and $J^{\prime}:=\{r \in S: M(r)=$ $s r \neq r s\} \subseteq J$. Then $u^{J} \in W_{S \backslash J^{\prime}}$ for all $u \leq w$.

Proof. Note first that $J^{\prime}=\{r \in J: r s \neq s r\}$. Let $u \in[e, w]$. Fix a reduced expression of $u^{J}$. Suppose, by contradiction, that $\left\{r \in S: r \leq u^{J}\right\} \cap J^{\prime} \neq \emptyset$. Consider the last letter of $r \in J^{\prime}$ appearing in this expression, say $r$. Then consider the first letter $t \notin J$ after $r$. Between $r$ and $t$ there cannot be any $s$ by Lemma 4.3.6. Hence there can only be letters commuting with $s$. By Lemma 4.4.3 after a finite number of steps we find a reduced expression of $u^{J}$ that ends with a letter in $J$ which is clearly a contradiction.

Proposition 4.4.5 Let $t \in S$ be such that $M$ is not a multiplication matching on $W_{\{s, t\}}(w)$. Suppose that $M(t)=t s$ and denote by $x_{0}=\alpha \beta \alpha \cdots$ tst the minimal element in $W_{\{s, t\}}(w)$ such that $M\left(x_{0}\right) \neq x_{0} s$. Then $\alpha \not \leq\left(u^{J}\right)^{\{s, t\}}$ for all $u \leq w$.

Proof. Consider a reduced expression for $u^{J}$ and a longest subsequence of this expression of the form $\alpha \beta \alpha \cdots t s t$, chosen with the left-most $\alpha$ and the rightmost $t$. Consider the first letter $r$ which appears after the first $\alpha$ distinct from $s$ and $t$. If $M(r)=r s \neq s r$ then this letter can be pushed on the left of the first $\alpha$ by Lemma 4.3.5. If $M(r)=r s=s r$ then, by Lemma 4.3.5, we are in one of the following three possibilities: $r$ commutes also with $t$, or it can be pushed on the left or it appears after the last $t$. In the first two cases it can be pushed on the left. So we can suppose that the first such letter $r$ appear after the last $t$. By Lemma 4.3.5, all the letters that appear after the last $t$ necessarily belong to $J$. So $u^{J}$ has a reduced expression in which after the first letter $\alpha$ there are only letters $s$ and $t$ and this clearly implies the statement.

Theorem 4.4.6 Let $(W, S)$ be a Coxeter system, $w \in W$ and $M$ be a special matching of $[e, w]$ with $M(e)=s$. Let $J:=\{r \in S: M(r)=s r\}$. Then
(i) If there exists a (necessarily unique) $t \in S$ such that $M(t)=t s$ but $M \not \equiv \rho_{s}$ on $W_{\{s, t\}}(w)$, then

$$
\left.M(u)=\left(u^{J}\right)^{\{s, t\}} M\left(\left(u^{J}\right)_{\{s, t\}}\left(u_{J}\right)_{\{s\}}\right)\right)^{\{s\}}\left(u_{J}\right),
$$

for all $u \leq w$.
(ii) If $M$ is a multiplication matching on $W_{\{x, s\}}$ for all $x \in S$, then

$$
M(u)=u^{J} s u_{J}
$$

for all $u \leq w$.
Proof. (i) We proceed by induction on $l(u)$ the result being clear if $l(u)=0$. Note that, by Proposition 4.3.3, $M\left(\left(u^{J}\right)_{\{s, t\}}\left(u_{J}\right)_{\{s\}}\right) \in W_{\{s, t\}}(w)$ and so, if we set

$$
u^{\prime}:=\left(u^{J}\right)^{\{s, t\}} M\left(\left(u^{J}\right)_{\{s, t\}}\left(u_{J}\right)_{\{s\}}\right)\{s\}\left(u_{J}\right),
$$

then $\left(u^{\prime J}\right)_{\{s, t\}}\left(u_{J}^{\prime}\right)_{\{s\}}=M\left(\left(u^{J}\right)_{\{s, t\}}\left(u_{J}\right)_{\{s\}}\right)$. We may assume that $M(u) \triangleright$ $u$ and that $M\left(\left(u^{J}\right)_{\{s, t\}}\left(u_{J}\right)_{\{s\}}\right) \triangleright\left(u^{J}\right)_{\{s, t\}}\left(u_{J}\right)_{\{s\}}$ otherwise we are done by induction.

Note first that if $u=\left(u^{J}\right)^{\{s, t\}}$ the result follows from Propositions 4.4.4 and 4.4.5 and Lemma 4.3.2, if $u=\left(u^{J}\right)_{\{s, t\}}\left(u_{J}\right)_{\{s\}}$ it is trivial, and if $u={ }^{\{s\}}\left(u_{J}\right)$ it follows from Lemma 4.3.2. Now consider the following three possibilities:

1. If $\left(u^{J}\right)^{\{s, t\}} \neq e$ let $x_{1} \in D_{L}\left(\left(u^{J}\right)^{\{s, t\}}\right)$ and $u_{1}:=x_{1} u$.
2. If $\left(u^{J}\right)_{\{s, t\}}\left(u_{J}\right)_{\{s\}} \neq e$ let $v \triangleleft\left(u^{J}\right)_{\{s, t\}}\left(u_{J}\right)_{\{s\}}$ be such that $M(v) \triangleright v$ and let $u_{2}:=\left(u^{J}\right)^{\{s, t\}} v^{\{s\}}\left(u_{J}\right)$.
3. If ${ }^{\{s\}}\left(u_{J}\right) \neq e$ let $x_{3} \in D_{R}\left({ }^{\{s\}}\left(u_{J}\right)\right)$ and $u_{3}:=u x_{3}$.

By our previous remark, we may certainly assume that at least two of these three hypotheses are satisfied and hence that there exists $i, j \in\{1,2,3\}, i \neq j$, such that $u_{i}$ and $u_{j}$ can be defined as above. Applying our induction hypothesis to $u_{i}$ and $u_{j}$ we have that $M\left(u_{i}\right) \triangleright u_{i}, M\left(u_{j}\right) \triangleright u_{j}$. The element $\left(u^{J}\right)^{\{s, t\}} M\left(\left(u^{J}\right)_{\{s, t\}}\left(u_{J}\right)_{\{s\}}\right){ }^{\{s\}}\left(u_{J}\right)$ covers $u, M\left(u_{i}\right)$ and $M\left(u_{j}\right)$. By Proposition 4.1.2, we conclude that $M(u)=\left(u^{J}\right)^{\{s, t\}} M\left(\left(u^{J}\right)_{\{s, t\}}\left(u_{J}\right)_{\{s\}}\right){ }^{\{s\}}\left(u_{J}\right)$.
(ii) This is similar and simpler than case (i) and is left to the reader.

Theorem 4.4.7 Let $(W, S)$ be a Coxeter system, $w \in W, l(w)>1$ and $M$ be a special matching of $[e, w]$ and suppose that $[e, w]$ is not a dihedral interval. Then there exists a multiplication matching $N$ of $[e, w]$ such that

1. $N(M(u))=M(N(u))$, for all $u \leq w$;

## 2. $N(w) \neq M(w)$.

Proof. Again we note that the result is true for a special matching $M$ if and only if it is true for $\tilde{M}$ and hence we can suppose that we are in one of the two cases of Theorem 4.4.6. Suppose to be in case (i). If $\left(w^{J}\right)^{\{s, t\}} \neq e$ let $x \in D_{L}\left(\left(w^{J}\right)^{\{s, t\}}\right)$. If $x \neq \beta$ then we choose $N=\lambda_{x}$. We have $M \equiv \rho_{s}$ on $W_{s, x}(w)$ by Proposition 5.14 and hence we are done by Lemma 4.2.5. If $x=\beta$ then there exists $r \in S, r<\left(w^{J}\right)^{\{s, t\}}$ such that $\beta r \neq r \beta$. Then if we let $K:=$ $\{r, s, t\}$, Proposition 4.3.10 applied to the interval $[e, w[K]]=W_{K}(w)$ implies that $M \lambda_{\beta}=\lambda_{\beta} M$ and the thesis follows by Lemma 4.2.5. If $\left(w^{J}\right)^{\{s, t\}}=e$ then necessarily ${ }^{\{s\}}\left(u_{J}\right) \neq e$ (otherwise $[e, w]$ is dihedral) and we proceed in a similar way considering a right descent of ${ }^{\{s\}}\left(u_{J}\right)$.

If we are in case (ii) the proof is left to the reader.

As a corollary of Theorem 4.4.7, we can prove Lusztig's conjecture on the lower Bruhat interval of any Coxeter system.

Corollary 4.4.8 Let $(W, S)$ be a Coxeter system, $w \in W$ and $M$ be a special matching of $[e, w]$. Then

$$
\widetilde{R}_{u, w}(q)=\widetilde{R}_{M(u), M(w)}(q)+\chi(M(u) \triangleright u) q \widetilde{R}_{u, M(w)}(q)
$$

for all $u \leq w$.
Proof. Straightforward by Theorems 4.4.1 and 4.4.7.

Corollary 4.4.9 Let $(W, S)$ and $\left(W^{\prime}, S^{\prime}\right)$ be two Coxeter systems, $w \in W$ and $w^{\prime} \in W^{\prime}$, and let e and $e^{\prime}$ be the identities of $W$ and $W^{\prime}$, respectively. Suppose that $\Phi:[e, w] \rightarrow\left[e^{\prime}, w^{\prime}\right]$ is an isomorphism of posets.
Then, for all $u, v \in[e, w]$, we have:

- $P_{u, v}(q)=P_{\Phi(u), \Phi(v)}(q)$,
- $R_{u, v}(q)=R_{\Phi(u), \Phi(v)}(q)$,
- $\widetilde{R}_{u, v}(q)=\widetilde{R}_{\Phi(u), \Phi(v)}(q)$.

Proof. Straightforward by Corollary 4.4.8.

### 4.5 Hecke algebra actions

In this section we introduce and study, for each $v \in W$, a Hecke algebra naturally associated to the special matchings of $[e, v]$ and an action of it on the submodule of the Hecke algebra of $W$ spanned by $\left\{T_{u}: u \leq v\right\}$. This action enables us to reformulate Corollary 4.4.8 in a very compact way in Theorem 4.5.2 by saying that this action "respects" the canonical involutions $\iota$ of these Hecke algebras. This, in turn, implies that the usual recursion for Kazhdan-Lusztig polynomials (Theorem 0.5.9) holds also when descents are replaced by special matchings (Corollary 4.5.4) thus giving a poset theoretic recursion for the Kazhdan-Lusztig polynomials which does not involve the $R$-polynomials.

Let $v \in W$ and $\mathcal{S}_{v}$ be the collection of all the special matchings of $[e, v]$. We denote by ( $\widehat{W}_{v}, \mathcal{S}_{v}$ ) the Coxeter system whose Coxeter generators are the elements of $\mathcal{S}_{v}$ and whose Coxeter matrix is given by $m(M, N):=o(M N)$, the period of $M N$ as a permutation of $[e, v]$. Then it is clear that we have a natural action of $\widehat{W}_{v}$ on the vector space $\oplus_{u \leq v} \mathbb{C}$. We denote by $\widehat{\mathcal{H}}_{v}$ the Hecke algebra of $\widehat{W}_{v}$ and by $\mathcal{H}_{v}$ the submodule of $\mathcal{H}$ defined by

$$
\mathcal{H}_{v}:=\bigoplus_{u \leq v} \mathbb{Z}\left[q^{\frac{1}{2}}, q^{-\frac{1}{2}}\right] T_{u} .
$$

Our first result defines the action of $\widehat{\mathcal{H}}_{v}$ on $\mathcal{H}_{v}$ that we wish to study. It is a natural generalization, and unification, of the left and right multiplication actions of $\mathcal{H}\left(W_{D_{L}(v)}\right)$ and $\mathcal{H}\left(W_{D_{R}(v)}\right)$ on $\mathcal{H}_{v}$.

Proposition 4.5.1 Let $v \in W$. Then there exists a unique action of $\widehat{\mathcal{H}}$ on $\mathcal{H}_{v}$ such that

$$
T_{M}\left(T_{u}\right)= \begin{cases}T_{M(u)}, & \text { if } M(u) \triangleright u,  \tag{4.4}\\ q T_{M(u)}+(q-1) T_{u}, & \text { otherwise },\end{cases}
$$

for all $u \leq v$ and all $M \in \mathcal{S}_{v}$.
Proof. The uniqueness part is trivial. To prove the existence we only have to check that $T_{M}\left(T_{M}\left(T_{u}\right)\right)=\left((q-1) T_{M}+q\right)\left(T_{u}\right)$ for all $u \leq v$ and $M \in \mathcal{S}_{v}$, and that, if $M, N \in \mathcal{S}_{v}$ and $m:=m(M, N)$, then

$$
\begin{equation*}
\underbrace{T_{M}\left(T _ { N } \left(T_{M}(\cdots\right.\right.}_{m}\left(T_{u}\right))))=\underbrace{T_{N}\left(T_{M}\left(T_{N}\left(\cdots\left(T_{u}\right)\right)\right)\right)}_{m} \tag{4.5}
\end{equation*}
$$

for all $u \leq v$. The proof of the first part is a simple verification and is left to
the reader.
To prove the second one let $M, N \in \mathcal{S}_{v}$ be such that $m(M, N)=m$ and $u \in[e, v]$. If $|\langle M, N\rangle(u)|=2 d$ then necessarily $d \mid m$. Let $I_{2}(d)$ be the dihedral group of order $2 d$ and $s$ and $t$, with $m(s, t)=d$, be its Coxeter generators. We define a poset isomorphism $\Phi:\langle M, N\rangle(u) \longrightarrow I_{2}(d)$ by

$$
\Phi(\underbrace{\cdots M N M}_{k}\left(u_{0}\right)):=\underbrace{\cdots s t s}_{k},
$$

for all $k \in[2 d]$, where $u_{0}$ is the smallest element in $\langle M, N\rangle(u)$, and extend this to a linear map $\Phi: \mathcal{H}(\langle M, N\rangle(u)) \longrightarrow \mathcal{H}\left(I_{2}(d)\right.$ ) (where $\mathcal{H}(\langle M, N\rangle(u)$ ) is the submodule of $\mathcal{H}_{v}$ spanned by $\left.\left\{T_{x}: x \in\langle M, N\rangle(u)\right\}\right)$ by $\Phi\left(T_{x}\right):=T_{\Phi(x)}$ for all $x \in\langle M, N\rangle(u)$. Then it is clear that $\Phi\left(T_{M} \circ T_{x}\right)=T_{s} \Phi\left(T_{x}\right)$ and $\Phi\left(T_{N} \circ T_{x}\right)=$ $T_{t} \Phi\left(T_{x}\right)$ for all $x \in\langle M, N\rangle(u)$. There follows that

$$
\begin{aligned}
\Phi(\underbrace{T_{M}\left(T _ { N } \left(T_{M}(\cdots\right.\right.}_{d}\left(T_{x}\right)))) & =\underbrace{T_{s} T_{t} T_{s} \cdots}_{d} \Phi\left(T_{x}\right) \\
& =\underbrace{T_{t} T_{s} T_{t} \cdots}_{d} \Phi\left(T_{x}\right) \\
& =\Phi(\underbrace{T_{N}\left(T _ { M } \left(T_{N}(\cdots\right.\right.}_{d}\left(T_{x}\right)))) .
\end{aligned}
$$

Hence $\underbrace{T_{M}\left(T_{N}\left(T_{M}(\cdots\right.\right.}_{d}\left(T_{x}\right))))=\underbrace{T_{N}\left(T_{M}\left(T_{N}(\cdots\right.\right.}_{d}\left(T_{x}\right))))$ for all $x \in\langle M, N\rangle(u)$ and (4.5) follows.

It is natural to wonder about the faithfulness of the action defined in (4.4). This will be adressed in Chapter 6.

We can now state and prove the first main result of this section, which is a compact reformulation of our main result (Corollary 4.4.8) in terms of the action of $\widehat{\mathcal{H}}_{v}$ on $\mathcal{H}_{v}$. Note that, by Proposition $0.5 .1, \mathcal{H}_{v}$ is invariant under the involution $\iota$ defined on $\mathcal{H}$. For convenience, we use the same symbol $\iota$ also for the corresponding involution of the Hecke algebra $\widehat{\mathcal{H}}_{v}$.

Theorem 4.5.2 Let $v \in W$. Then for all $h \in \mathcal{H}_{v}, \hat{h} \in \widehat{\mathcal{H}}_{v}$

$$
\iota(\hat{h}(h))=\iota(\hat{h})(\iota(h)) .
$$

Proof. We may clearly assume that $h=T_{u}$ for some $u \leq v$ and $\hat{h}=T_{M}$, where
$M$ is a special matching of $[e, v]$.
Suppose first that $u \triangleleft M(u)$. Then, by (4.4) and Proposition 0.5.1, we have

$$
\iota\left(T_{M}\left(T_{u}\right)\right)=\iota\left(T_{M(u)}\right)=\left(T_{M(u)^{-1}}\right)^{-1}=-\varepsilon_{u} q^{-l(u)-1} \sum_{x} \varepsilon_{x} R_{x, M(u)} T_{x}
$$

where $\varepsilon_{y}=(-1)^{l(y)}$ for all $y \in W$.
On the other hand

$$
\begin{aligned}
\iota\left(T_{M}\right)\left(\iota\left(T_{u}\right)\right)= & T_{M}^{-1}\left(T_{u^{-1}}^{-1}\right) \\
= & {\left[q^{-1} T_{M}-\left(1-q^{-1}\right)\right]\left(\varepsilon_{u} q^{-l(u)} \sum_{x} \varepsilon_{x} R_{x, u} T_{x}\right) } \\
= & \varepsilon_{u} q^{-l(u)}\left\{\sum_{x \triangleleft M(x)}\left[q^{-1} \varepsilon_{x} R_{x, u} T_{M(x)}-\left(1-q^{-1}\right) \varepsilon_{x} R_{x, u} T_{x}\right]+\right. \\
& \left.\sum_{x \triangleright M(x)}\left[q^{-1} \varepsilon_{x} R_{x, u}\left(q T_{M(x)}+(q-1) T_{x}\right)-\left(1-q^{-1}\right) \varepsilon_{x} R_{x, u} T_{x}\right]\right\} \\
= & -\varepsilon_{u} q^{-l(u)}\left[\sum_{M(x) \triangleleft x} q^{-1} \varepsilon_{x} R_{M(x), u} T_{x}+\right. \\
& \left.\sum_{M(x) \triangleright x}\left(1-q^{-1}\right) \varepsilon_{x} R_{x, u} T_{x}+\sum_{M(x) \triangleright x} \varepsilon_{x} R_{M(x), u} T_{x}\right] \\
= & -\varepsilon_{u} q^{-l(u)}\left[\sum_{M(x) \triangleleft x} q^{-1} \varepsilon_{x} R_{x, M(u)} T_{x}+\sum_{x \triangleleft M(x)} q^{-1} \varepsilon_{x} R_{x, M(u)} T_{x}\right]
\end{aligned}
$$

by Corollary 4.4.8 and the assertion follows in this case.
Suppose now that $u \triangleright M(u)$. Then applying what we have just proved to $M(u)$ yields that

$$
T_{u^{-1}}^{-1}=\iota\left(T_{u}\right)=\iota\left(T_{M}\left(T_{M(u)}\right)\right)=\iota\left(T_{M}\right)\left(\iota\left(T_{M(u)}\right)\right)=T_{M}^{-1}\left(T_{M(u)^{-1}}^{-1}\right)
$$

Therefore, by Proposition 4.4, $T_{M}\left(T_{u^{-1}}^{-1}\right)=T_{M(u)^{-1}}^{-1}$. Hence

$$
\begin{aligned}
\iota\left(T_{M}\left(T_{u}\right)\right) & =\iota\left(q T_{M(u)}+(q-1) T_{u}\right) \\
& =q^{-1} T_{M(u)^{-1}}^{-1}+\left(q^{-1}-1\right) T_{u^{-1}}^{-1} \\
& =q^{-1} T_{M}\left(T_{u^{-1}}^{-1}\right)+\left(q^{-1}-1\right) T_{u^{-1}}^{-1} \\
& =\left[q^{-1} T_{M}-\left(1-q^{-1}\right)\right]\left(T_{u^{-1}}^{-1}\right) \\
& =T_{M}^{-1}\left(T_{u^{-1}}^{-1}\right)
\end{aligned}
$$

$$
=\iota\left(T_{M}\right)\left(\iota\left(T_{u}\right)\right),
$$

and the result again follows.

Recall from Theorem 0.5.4 the definition of the Kazhdan-Lusztig basis $C^{\prime}=$ $\left\{C_{v}^{\prime}: v \in W\right\}$ of the Hecke algebra of $W$.

Theorem 4.5.3 Let $v \in W$ and $M \in \mathcal{S}_{v}$. Then, for all $x \in[e, v]$,

$$
C_{M}^{\prime}\left(C_{x}^{\prime}\right)= \begin{cases}C_{M(x)}^{\prime}+\sum_{\{z: M(z) \triangleleft z\}} \mu(z, x) C_{z}^{\prime}, & \text { if } M(x) \triangleright x, \\ \left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right) C_{x}^{\prime}, & \text { if } M(x) \triangleleft x,\end{cases}
$$

in $\mathcal{H}_{v}$.

Proof. Suppose first that $M(x) \triangleright x$. Let, for brevity, $D_{M(x)}:=C_{M}^{\prime}\left(C_{x}^{\prime}\right)-$ $\sum_{\{z: M(z) \triangleleft z\}} \mu(z, x) C_{z}^{\prime}$. To prove that $D_{M(x)}=C_{M(x)}^{\prime}$ we use the characterization of the Kazhdan-Lusztig basis given in Theorem 0.5.4 ([39, Theorem 7.9]). It is clear from Theorem 4.5.2 that $\iota\left(D_{M(x)}\right)=D_{M(x)}$. So we only need to show that if

$$
D_{M(x)}=q^{-\frac{l(M(x))}{2}} \sum_{u \leq M(x)} \widetilde{P}_{u, M(x)}(q) T_{u}
$$

then
i) $\widetilde{P}_{M(x), M(x)}(q)=1$,
ii) $\widetilde{P}_{u, M(x)}(q) \in \mathbb{Z}[q]$ and has degree $<\frac{1}{2} l(u, M(x))$ if $u<M(x)$.

We distinguish two cases.
Suppose $u \triangleleft M(u)$. Then $T_{M}\left(C_{x}^{\prime}\right)$ involves $T_{u}$ with coefficient $q^{-\frac{l(x)}{2}} q P_{M(u), x}(q)$. It follows easily that the coefficient of $T_{u}$ in $C_{M}^{\prime}\left(C_{x}^{\prime}\right)$ is

$$
q^{-\frac{l(M(x))}{2}} q P_{M(u), x}(q)+q^{-\frac{l(M(x))}{2}} P_{u, x}(q) .
$$

On the other hand, if $u \triangleright M(u), T_{M}\left(C_{x}^{\prime}\right)$ involves $T_{u}$ with coefficient equal to $q^{-\frac{l(x)}{2}}\left(P_{M(u), x}(q)+(q-1) P_{u, x}(q)\right)$. Again it follows easily that the coefficient of $T_{u}$ in $C_{M}^{\prime}\left(C_{x}^{\prime}\right)$ is

$$
q^{-\frac{l(M(x))}{2}} P_{M(u), x}(q)+q^{-\frac{l(M(x))}{2}} q P_{u, x}(q) .
$$

Finally, the coefficient of $T_{u}$ in $\sum \mu(z, x) C_{z}^{\prime}$ is in both cases

$$
\sum_{\{z: M(z) \triangleleft z\}} \mu(z, x) q^{-\frac{l(z)}{2}} P_{u, z}(q) .
$$

So, if we set $c=1$ if $M(u) \triangleleft u$ and $c=0$ otherwise, we only have to show that the polynomials

$$
q^{1-c} P_{M(u), x}(q)+q^{c} P_{u, x}(q)-\sum_{\{z: M(z) \triangleleft z\}} \mu(z, x) q^{\frac{l(z, M(x))}{2}} P_{u, z}(q)
$$

satisfy properties i) and ii). The proof of this fact can be done in exactly the same way as the proof of [39, Theorem 7.9] (see [39, § 7.11]) and it is therefore omitted.

Assume now that $M(x) \triangleleft x$. We proceed by induction on $l(x)$. If $l(x)=1$ then necessarily $x=M(e)$ and the result is easy to verify. So assume $l(x) \geq 2$. Then by what we have just proved we have that

$$
\begin{equation*}
C_{x}^{\prime}=C_{M}^{\prime}\left(C_{M(x)}^{\prime}\right)-\sum_{\{z: M(z) \triangleleft z\}} \mu(z, M(x)) C_{z}^{\prime} \tag{4.6}
\end{equation*}
$$

Therefore, since $C_{M}^{\prime} C_{M}^{\prime}=\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right) C_{M}^{\prime}$,

$$
\begin{aligned}
C_{M}^{\prime}\left(C_{x}^{\prime}\right) & =\left(C_{M}^{\prime} C_{M}^{\prime}\right)\left(C_{M(x)}^{\prime}\right)-\sum_{\{z: M(z) \triangleleft z\}} \mu(z, M(x)) C_{M}^{\prime}\left(C_{z}^{\prime}\right) \\
& =\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right) C_{x}^{\prime},
\end{aligned}
$$

by (4.6) and our induction hypothesis, as desired.
Theorem 4.5.3, and its proof, imply the following poset theoretic recursion for Kazhdan-Lusztig polynomials depending on special matchings. It generalizes the usual recursion for Kazhdan-Lusztig polynomials depending on left or right descents (Theorem 0.5.9).

Corollary 4.5.4 Let $u, v \in W, u<v$, and $M$ be a special matching of $[e, v]$. Then
$P_{u, v}(q)=q^{1-c} P_{M(u), M(v)}(q)+q^{c} P_{u, M(v)}(q)-\sum_{\{z: M(z) \triangleleft z\}} \mu(z, M(v)) q^{\frac{l(z, v)}{2}} P_{u, z}(q)$
where $c=1$ if $M(u) \triangleleft u$ and $c=0$ otherwise.

We illustrate Corollary 4.5.4 with an example. Let $v=3421 \in \mathfrak{S}(4)$. The Bruhat interval $[e, v]$ has 5 distinct special matchings, $l_{2}, \rho_{2}, \rho_{3}, \lambda_{2}, \lambda_{1}$, which are shown in Figure 4.6 (for the reason of the notation $l_{2}$ see Theorem 0.7.3). Using Corollary 4.5.4 for the special matching $l_{2}$ we obtain

$$
\begin{aligned}
P_{e, v} & =q P_{l_{2}(e), l_{2}(v)}+P_{e, l_{2}(v)}-\sum_{\left\{z: l_{2}(z) \triangleleft z\right\}} \mu\left(z, l_{2}(v)\right) q^{\frac{l(z, v)}{2}} P_{e, z} \\
& =q P_{1324,3412}+P_{e, 3412}-\left(1 \cdot q \cdot P_{e, 1432}+1 \cdot q \cdot P_{e, 3214}+1 \cdot q^{2} \cdot P_{e, 1324}\right) \\
& =q(q+1)+(q+1)-q-q-q^{2} .
\end{aligned}
$$

Note that using the other 4 special matchings we obtain genuinely different computations for $P_{e, 3421}$. In fact, we obtain

$$
P_{e, 3421}= \begin{cases}q+1-q & \text { using } \rho_{2}, \\ q+(1+q)-q-q & \text { using } \rho_{3} \\ q+1-q & \text { using } \lambda_{2} \\ q+(1+q)-q-q & \text { using } \lambda_{1}\end{cases}
$$

The reason for this is that the special matching $l_{2}$ is not isomorphic to any other special matching of [ $e, 3421$ ], namely that do not exist a poset- automorphism $\Phi$ of $[e, 3421]$ and a special matching $M$ of $[e, 3421]$ satisfying $\Phi l_{2}(x)=M \Phi(x)$. In fact, any automorphism $\Phi$ of [ $e, 3421$ ] must fix 1324 and 3412 , namely $\Phi(1324)=$ 1324 and $\Phi(3412)=3412$. Therefore, any special matching $M$ of $[e, v]$ such that $\Phi \circ M=l_{2} \circ \Phi$ must satisfy $M(e)=1324$ and $M(3421)=3412$, but $l_{2}$ is the unique special matching of $[e, v]$ satisfying these two conditions. Actually, more is true. Suppose that $u \in \mathfrak{S}(n)$ is such that $[e, u] \cong[e, 3421]$ (posetisomorphism). Since $[e, v]$ has only three atoms we deduce that any reduced expression of $u$ is composed of letters of exactly 3 different kinds, say $s_{i}, s_{j}$ and $s_{k}$, with $i<j<k$. If these indices are not consecutive we would have at most 4 permutations of length 2 in $[e, u]$. So the indices must be consecutive and we may assume that $s_{i}=s_{1}, s_{j}=s_{2}, s_{k}=s_{3}$ and $u \in \mathfrak{S}(4)$. But in $\mathfrak{S}(4)$ there are only 3 permutations of length 5, namely $v, v^{-1}$ and 4231, and [e, 4231] has 4 coatoms. Hence the special matching $l_{2}$ of $[e, 3421]$ is not isomorphic to any multiplication matching in any symmetric group. In fact, with more work one can show that the special matching $l_{2}$ of [ $e, 3421$ ] is not isomorphic to any multiplication matching in any Coxeter system (even infinite). We leave this to the interested reader.


Figure 4.6: The special matchings of $[e, 3421]$.

## Chapter 5

## Combinatorial poset theoretic formulae

In this chapter, we introduce three families of sequences of special matchings: the regular sequences, the $B$-regular sequences, and the $R$-regular sequences. All of them are new combinatorial analogues of the concept of reduced expression. Using these sequences, we find some formulae valid for Kazhdan-Lusztig and $R$-polynomials of any Coxeter system.

### 5.1 Regular sequences

Our purpose in this section is to generalize an algorithm and a closed formula of Deodhar ([28, Algorithm 4.11] and [26, Theorem 1.3]) for Kazhdan-Lusztig and $R$-polynomials, respectively.

Let $v \in W$. We say that a sequence $\left(M_{1}, \ldots, M_{l}\right)$ (where $\left.l:=l(v)\right)$ is a regular sequence (of special matchings) for $v$ if, for all $i=1, \ldots, l, M_{i}$ is a special matching of $\left[e, M_{i+1} \cdots M_{l}(v)\right]$. Note that, in particular, $M_{1} \cdots M_{l}(v)=$ $e$. The regular chain associated to a regular sequence $\left(M_{1}, \ldots, M_{l}\right)$ for $v$ is $\left(v_{0}, \ldots, v_{l}\right)$ where $v_{i}:=M_{i+1} \cdots M_{l}(v)=M_{i} \cdots M_{1}(e)$, for $i=0, \ldots, l$. Clearly, $e=v_{0} \triangleleft v_{1} \triangleleft \cdots \triangleleft v_{l}=v$ and $M_{i}\left(v_{i-1}\right)=v_{i}$, for $i=1, \ldots, l$.

For example, if $W=\mathfrak{S}(4)$ and $v=3421$ then the sequence $\left(\lambda_{1}, \rho_{3}, \lambda_{2}, \rho_{2}, l_{2}\right)$ illustrated in Figure 4.6 is a regular sequence for $v$. Note that, if $s_{i_{1}} \cdots s_{i_{l}}$ is a reduced expression for $v$, then $\left(\lambda_{i_{l}}, \ldots, \lambda_{i_{1}}\right)$ and $\left(\rho_{i_{1}}, \ldots, \rho_{i_{l}}\right)$ are two regular sequences for $v$. Thus, the concept of a regular sequence is a generalization of
that of a reduced expression. We say that a regular sequence $\mathcal{M}=\left(M_{1}, \ldots, M_{l}\right)$ for $v$ comes from a reduced expression if there is a reduced expression $s_{i_{1}} \cdots s_{i_{l}}$ of $v$ such that either $\mathcal{M}=\left(\lambda_{i_{l}}, \ldots, \lambda_{i_{1}}\right)$ or $\mathcal{M}=\left(\rho_{i_{1}}, \ldots, \rho_{i_{l}}\right)$.

Our first results are the analogues, for any regular sequence, of two well known results for reduced expressions. They are used repeatedly throughout the rest of this work, often without explicit mention.

Lemma 5.1.1 Let $v \in W$, and $\left(M_{1}, \ldots, M_{l}\right)$ be a regular sequence for $v$. Then for all $u \leq v$ there exists $1 \leq i_{1}<\ldots<i_{k} \leq l$ such that $\left(M_{i_{1}}, \ldots, M_{i_{k}}\right)$ is a regular sequence for $u$.

Proof. We proceed by induction on $l$ the statement being trivial for $l=1$. So assume that $l>1$. Note that $\left(M_{1}, \ldots, M_{l-1}\right)$ is a regular sequence for $M_{l}(v)$. Let $u \in[e, v]$. If $M_{l}(u) \triangleleft u$ then, by Lemma 0.7.1, $M_{l}(u) \leq M_{l}(v)$ so by induction there exist $1 \leq i_{1}<\ldots<i_{k} \leq l-1$ such that $\left(M_{i_{1}}, \ldots, M_{i_{k}}\right)$ is a regular sequence for $M_{l}(u)$, hence $\left(M_{i_{1}}, \ldots, M_{i_{k}}, M_{l}\right)$ is a regular sequence for $u$. If $M_{l}(u) \triangleright u$ then, by Lemma $0.7 .1, u \leq M_{l}(v)$ and we conclude again by induction.

As a corollary of the previous result, we obtain a generalization of the Exchange Property (Theorem 0.3.1).

Corollary 5.1.2 Let $v \in W$ and $\left(M_{1}, \ldots, M_{l}\right)$ be a regular sequence for $v$. Let $M$ be a special matching of $v$. Then there exists $i \in[l]$ such that

$$
M(v)=M M_{l} \cdots M_{1}(e)=M_{l} \cdots \widehat{M}_{i} \cdots M_{1}(e)
$$

where $\widehat{M}_{i}$ means that $M_{i}$ has been deleted.
Proof. By Lemma 5.1.1, there exists a subsequence of $\left(M_{1}, \ldots, M_{l}\right)$ which is a regular sequence for $M(v)$.

Lemma 5.1.3 Let $v \in W$ and $\left(M_{1}, \ldots, M_{l}\right)$ be a regular sequence for $v$. Then the composition $M_{i_{k}} \cdots M_{i_{1}}(e)$ is defined for any $1 \leq i_{1}<i_{2}<\cdots<i_{k} \leq l$.

Proof. Let $\left(v_{0}, \ldots, v_{l}\right)$ be the regular chain associated to $\left(M_{1}, \ldots, M_{l}\right)$. We will show that $M_{i_{k}} \cdots M_{i_{1}}(e)$ is defined and $M_{i_{k}} \cdots M_{i_{1}}(e) \leq v_{i_{k}}$ for all $1 \leq$ $i_{1}<i_{2}<\cdots<i_{k} \leq l$.

We proceed by induction on $k$, the claim being clear if $k=0$. So let $1 \leq i_{1}<$ $i_{2}<\cdots<i_{k} \leq l$, with $k \geq 1$. By our induction hypothesis $u:=M_{i_{k-1}} \cdots M_{i_{1}}(e)$ is defined and $u \leq v_{i_{k-1}}<v_{i_{k}}$. But, by the definition of a regular sequence of
special matchings, $M_{i_{k}}$ is a special matching of $\left[e, v_{i_{k}}\right]$. Therefore $M_{i_{k}}(u)$ is defined and $M_{i_{k}}(u) \leq v_{i_{k}}$, as desired.

Let $v \in W$ and $\mathcal{M}=\left(M_{1}, \ldots, M_{l}\right)$ be a regular sequence for $v$ (so $\left.l=l(v)\right)$. Given $S=\left\{i_{1}, \ldots, i_{k}\right\}<\subseteq[l]$ we let

$$
\pi(S):=M_{i_{k}} \cdots M_{i_{1}}(e)
$$

and we define, for each $j \in[l]$,

$$
\varepsilon_{j}(S):= \begin{cases}1, & \text { if } M_{j}(y) \triangleleft y \\ 0, & \text { if } M_{j}(y) \triangleright y\end{cases}
$$

where $y:=\pi(S \cap[j-1])$. We also let

$$
d_{1}(S, l):=\sum_{j \in[l] \backslash S} \varepsilon_{j}(S)
$$

and

$$
d_{2}(S):=\sum_{j \in S} \varepsilon_{j}(S)
$$

Note that $\left(M_{i_{1}}, \ldots, M_{i_{k}}\right)$ is a regular sequence for $M_{i_{k}} \cdots M_{i_{1}}(e)$ if and only if $d_{2}(S)=0$. Let, for brevity,

$$
d(S, l):=d_{1}(S, l)+d_{2}(S)
$$

We say that $S$ is distinguished, with respect to $\mathcal{M}$, if $d_{1}(S, l)=0$. In the case that $\mathcal{M}$ comes from a reduced expression this concept coincides with the one introduced by Deodhar in [26, Def. 2.3]. We denote by $\mathcal{D}(\mathcal{M})$ the set of all subsets of $[l]$ which are distinguished with respect to $\mathcal{M}$, and we let, for $u \in W$,

$$
\mathcal{D}(\mathcal{M})_{u}:=\{S \in \mathcal{D}(\mathcal{M}): \pi(S)=u\}
$$

We can now prove the first main result of this section. It is a closed formula for $R$-polynomials which generalizes Theorem 1.3 of [26].

Theorem 5.1.4 Let $v \in W$ and $\mathcal{M}=\left(M_{1}, \ldots, M_{l}\right)$ be a regular sequence for $v$. Then

$$
\widetilde{R}_{u, v}(q)=\sum_{S \in \mathcal{D}(\mathcal{M})_{u}} q^{l(v)-|S|},
$$

for all $u \in W$.

Proof. Our proof is similar to the one given in [26, §5], but simpler. The result is clear if $u \not \leq v$, so assume $u \leq v$. We proceed by induction on $l:=l(v)$, the result being trivial if $l=0$.

So assume $l \geq 1$ and let, for convenience, $M:=M_{l}$. We distinguish two cases.
a) $M(u) \triangleleft u$.

This implies that if $S \in \mathcal{D}(\mathcal{M})_{u}$ then $l \in S$ by the definition of a distinguished subset. Note that $\left(M_{1}, \ldots, M_{l-1}\right)$ is a regular sequence for $M(v)$. Define a map

$$
\varphi: \mathcal{D}(\mathcal{M})_{u} \longrightarrow \mathcal{D}\left(M_{1}, \ldots, M_{l-1}\right)_{M(u)}
$$

by letting $\varphi(S)=S \backslash\{l\}$ for all $S \in \mathcal{D}(\mathcal{M})_{u}$. The map $\varphi$ is well-defined and bijective since $l \in S$. Therefore, by Corollary 4.4.8 and our induction hypothesis

$$
\sum_{S \in \mathcal{D}(\mathcal{M})_{u}} q^{l(v)-|S|}=\sum_{S^{\prime} \in \mathcal{D}\left(M_{1}, \ldots, M_{l-1}\right)_{M(u)}} q^{l(M(v))-\left|S^{\prime}\right|}=\widetilde{R}_{M(u), M(v)}(q)=\widetilde{R}_{u, v}(q) .
$$

b) $M(u) \triangleright u$.

Let $\mathcal{D}(\mathcal{M})_{u}^{-}:=\left\{S \in \mathcal{D}(\mathcal{M})_{u}: l \notin S\right\}$ and $\mathcal{D}(\mathcal{M})_{u}^{+}:=\left\{S \in \mathcal{D}(\mathcal{M})_{u}: l \in S\right\}$. Define a map $\varphi: \mathcal{D}(\mathcal{M})_{u} \longrightarrow \mathcal{D}\left(M_{1}, \ldots, M_{l-1}\right)_{u} \cup \mathcal{D}\left(M_{1}, \ldots, M_{l-1}\right)_{M(u)}$ by letting

$$
\varphi(S)= \begin{cases}S, & \text { if } l \notin S \\ S \backslash\{l\}, & \text { if } l \in S\end{cases}
$$

for all $S \in \mathcal{D}(\mathcal{M})_{u}$.
We claim that $\varphi$ is a bijection, that $\varphi\left(\mathcal{D}(\mathcal{M})_{u}^{-}\right)=\mathcal{D}\left(M_{1}, \ldots, M_{l-1}\right)_{u}$ and that $\varphi\left(\mathcal{D}(\mathcal{M})_{u}^{+}\right)=\mathcal{D}\left(M_{1}, \ldots, M_{l-1}\right)_{M(u)}$. All verifications are obvious, except for the surjectivity of $\varphi$. But if $S^{\prime} \in \mathcal{D}\left(M_{1}, \ldots M_{l-1}\right)_{u}$ then $S^{\prime} \in \mathcal{D}(\mathcal{M})_{u}$ (since $M(u) \triangleright u)$, and if $S^{\prime \prime} \in \mathcal{D}\left(M_{1}, \ldots, M_{l-1}\right)_{M(u)}$ then $S^{\prime \prime} \cup\{l\} \in \mathcal{D}(\mathcal{M})_{u}$ and this proves the surjectivity. Therefore, by Corollary 4.4.8 and our induction hypothesis,

$$
\begin{aligned}
\sum_{S \in \mathcal{D}(\mathcal{M})_{u}} q^{l(v)-|S|}= & \sum_{S^{\prime} \in \mathcal{D}\left(M_{1}, \ldots, M_{l-1}\right)_{u}} q^{l(M(v))-\left|S^{\prime}\right|+1}+ \\
& \sum_{S^{\prime \prime} \in \mathcal{D}\left(M_{1}, \ldots, M_{l-1}\right)_{M(u)}} q^{l(M(v))-\left|S^{\prime \prime}\right|} \\
= & q \widetilde{R}_{u, M(v)}(q)+\widetilde{R}_{M(u), M(v)}(q)
\end{aligned}
$$

$$
=\widetilde{R}_{u, v}(q),
$$

as desired.
The preceding result has the following consequence, which is needed in the rest of this section, and which appears to be difficult to prove directly.

Corollary 5.1.5 Let $v \in W$ and $\left(M_{1}, \ldots, M_{l}\right)$ be a regular sequence for $v$. Then $\pi$ is a bijection between $\left\{S \subseteq[l]: d_{1}(S, l)=d_{2}(S)=0\right\}$ and $[e, v]$.

Proof. Clearly, $\pi(S) \in[e, v]$. Furthermore, since $\left[t^{l(u, v)}\right]\left(\widetilde{R}_{u, v}\right)=1$ for all $u \in[e, v]$, we conclude from Theorem 5.1.4 that for each $u \in[e, v]$ there exists a unique distinguished subset $S_{u}$ such that $\pi\left(S_{u}\right)=u$. Since a subset $S \subseteq[l]$ is distinguished if and only if $d_{1}(S, l)=0$, and since $l(\pi(S))=|S|$ if and only if $d_{2}(S)=0$, the result follows.

In order to prove the second main result of this section we need some further properties of the action of the Hecke algebra $\widehat{\mathcal{H}}_{v}$ on the module $\mathcal{H}_{v}$ defined in Section 4.5. The next result is the analogue, for regular sequences, of Proposition 3.5 of [28].

Proposition 5.1.6 Let $v \in W$ and $\left(M_{1}, \ldots, M_{l}\right)$ be a regular sequence for $v$. Then

$$
\begin{equation*}
q^{\frac{l}{2}} C_{M_{l}}^{\prime}\left(C_{M_{l-1}}^{\prime}\left(\cdots\left(C_{M_{1}}^{\prime}\left(T_{e}\right)\right)\right)\right)=\sum_{S \subseteq[l]} q^{d(S, l)} T_{\pi(S)}, \tag{5.1}
\end{equation*}
$$

in $\mathcal{H}_{v}$.

Proof. Let, for brevity, $C_{i}^{\prime}:=C_{M_{i}}^{\prime}$ and $T_{i}:=T_{M_{i}}$ for $i=1, \ldots, l$. Note first that the left-hand side of (5.1) is well defined since $C_{i}^{\prime}, T_{i} \in \widehat{\mathcal{H}}_{v_{i}}$, for $i=1, \ldots, l$, (where $\left(v_{0}, \ldots, v_{l}\right)$ is the regular chain associated to $\left(M_{1}, \ldots, M_{l}\right)$ ). We proceed by induction on $l \geq 1$, (5.1) being clear if $l=1$.
So let $l \geq 2$ and suppose that (5.1) holds for $l-1$. Recall that $C_{i}^{\prime}=q^{-\frac{1}{2}}\left(1+T_{i}\right)$. Then we have

$$
\begin{aligned}
& q^{\frac{l}{2}} C_{l}^{\prime}\left(\left(C _ { l - 1 } ^ { \prime } \left(\left(\cdots\left(\left(C_{1}^{\prime}\left(T_{e}\right)\right)\right)\right)=\left(1+T_{l}\right)\left(\sum_{S \subseteq[l-1]} q^{d(S, l-1)} T_{\pi(S)}\right)\right.\right.\right. \\
& \quad=\sum_{S \subseteq[l-1]} q^{d(S, l-1)} T_{\pi(S)}
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{\{S \subseteq[l-1]:} \sum_{\left.M_{l}(\pi(S)) \triangleright \pi(S)\right\}} q^{d(S, l-1)} T_{M_{l}(\pi(S))} \\
& +\sum_{\{S \subseteq[l-1]:} \sum_{\left.M_{l}(\pi(S)) \triangleleft \pi(S)\right\}} q^{d(S, l-1)}\left(q T_{M_{l}(\pi(S))}+(q-1) T_{\pi(S)}\right) \\
= & \sum_{\left\{S \subseteq[l-1]: M_{l}(\pi(S)) \triangleright \pi(S)\right\}} q^{d(S, l-1)} T_{\pi(S)} \\
& +\sum_{\{S \subseteq[l-1]:} \sum_{\left.M_{l}(\pi(S)) \triangleright \pi(S)\right\}} q^{d(S, l-1)} T_{\pi(S \cup\{l\})} \\
& +\sum_{\left\{S \subseteq[l-1]: M_{l}(\pi(S)) \triangleleft \pi(S)\right\}} q^{d(S, l-1)+1}\left(T_{\pi(S \cup\{l\})}+T_{\pi(S)}\right) \\
= & \sum_{S \subseteq[l-1]} q^{d(S, l)} T_{\pi(S)}+\sum_{S \subseteq[l-1]} q^{d(S \cup\{l\}, l)} T_{\pi(S \cup\{l\})},
\end{aligned}
$$

since $d(S, l)=d(S \cup\{l\}, l)=d(S, l-1)+\varepsilon_{l}(S)$, and (5.1) follows.
For brevity, we call a Coxeter system $(W, S)$ nonnegative if its KazhdanLusztig polynomials $P_{u, v}$ have nonnegative coefficients for all $u, v \in W$.

Proposition 5.1.7 Let $(W, S)$ be a nonnegative Coxeter system, $v \in W$, and $\left(M_{1}, \ldots, M_{l}\right)$ be a regular sequence for $v$. Then there exist $L_{x} \in \mathbb{N}\left[q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right]$, for each $x \leq v$, such that $L_{v}=1$ and

$$
\begin{equation*}
C_{M_{l}}^{\prime}\left(C_{M_{l-1}}^{\prime}\left(\cdots\left(C_{M_{1}}^{\prime}\left(T_{e}\right)\right)\right)\right)=\sum_{x \leq v} L_{x} C_{x}^{\prime} . \tag{5.2}
\end{equation*}
$$

Proof. Let, for brevity, $C_{i}^{\prime}:=C_{M_{i}}^{\prime}$ for $i=1, \ldots, l$. We proceed by induction on $l \geq 1$, (5.2) being clear if $l=1$ (with $L_{e}=0$ ).

So let $l \geq 2$ and suppose that (5.2) holds for $l-1$. Then there exists $\tilde{L}_{x} \in \mathbb{N}\left[q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right]$ for each $x \leq M_{l}(v)$ such that

$$
C_{l-1}^{\prime}\left(C_{l-2}^{\prime}\left(\cdots\left(C_{1}^{\prime}\left(T_{e}\right)\right)\right)\right)=\sum_{x \leq M_{l}(v)} \tilde{L}_{x} C_{x}^{\prime}
$$

and $\tilde{L}_{M_{l}(v)}=1$. Therefore, by Theorem 4.5.3,

$$
\begin{aligned}
C_{l}^{\prime}\left(C_{l-1}^{\prime}\left(\cdots\left(C_{1}^{\prime}\left(T_{e}\right)\right)\right)\right)= & C_{l}^{\prime}\left(\sum_{x \leq M_{l}(v)} \tilde{L}_{x} C_{x}^{\prime}\right) \\
= & \sum_{\left\{x \leq M_{l}(v): M_{l}(x) \triangleright x\right\}} \tilde{L}_{x}\left[C_{M_{l}(x)}^{\prime}+\sum_{\left\{z: M_{l}(z) \triangleleft z\right\}} \mu(z, x) C_{z}^{\prime}\right] \\
& +\sum_{\left\{x \leq M_{l}(v): M_{l}(x) \triangleleft x\right\}}\left(q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right) \tilde{L}_{x} C_{x}^{\prime},
\end{aligned}
$$

and the result follows.
We can now prove the second main result of this section, which plays a fundamental role in the algorithm.

Theorem 5.1.8 Given a nonnegative Coxeter system $(W, S)$ and $v \in W$, let $\left(M_{1}, \ldots, M_{l}\right)$ be a regular sequence for $v$, and $A \subseteq\left\{x \in[e, v]: L_{x} \neq 0\right\}, v \in A$. Then there esists $\mathcal{E} \subseteq \mathcal{P}([l])$ such that

$$
\begin{equation*}
q^{-\frac{l}{2}} \sum_{S \in \mathcal{E}} q^{d(S, l)} T_{\pi(S)}=\sum_{x \in A} L_{x} C_{x}^{\prime} \tag{5.3}
\end{equation*}
$$

Furthermore, for any $y \in A \backslash\{v\}$, $y$ is maximal in $A \backslash\{v\}$ if and only if

$$
\begin{equation*}
\operatorname{deg}\left(\sum_{\{S \in \mathcal{E}: \pi(S)=y\}} q^{d(S, l)}\right) \geq \frac{l(y, v)}{2} \tag{5.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{deg}\left(\sum_{\{S \in \mathcal{E}: \pi(S)=x\}} q^{d(S, l)}\right)<\frac{l(x, v)}{2} \tag{5.5}
\end{equation*}
$$

for all $y<x<v$. If these conditions are satisfied then

$$
\begin{equation*}
L_{y}=\sum_{\left\{S \in \mathcal{E}: \pi(S)=y, d(S, l) \geq \frac{l(y, v)}{2}\right\}} q^{d(S, l)-\frac{l(y, v)}{2}}+\sum_{\left\{S \in \mathcal{E}: \pi(S)=y, d(S, l)>\frac{l(y, v)}{2}\right\}} q^{\frac{l(y, v)}{2}-d(S, l)} \tag{5.6}
\end{equation*}
$$

and

$$
\begin{equation*}
P_{y, v}=\sum_{\left\{S \in \mathcal{E}: \pi(S)=y, d(S, l)<\frac{l(y, v)}{2}\right\}} q^{d(S, l)}-\sum_{\left\{S \in \mathcal{E}: \pi(S)=y, d(S, l)>\frac{l(y, v)}{2}\right\}} q^{l(y, v)-d(S, l)} . \tag{5.7}
\end{equation*}
$$

Proof. Let $x \in[e, v]$. The coefficient of $T_{x}$ in the right-hand side of (5.3) is $\sum_{y \in A} L_{y} q^{-\frac{l(y)}{2}} P_{x, y}$. Since, by our hypotheses, $L_{y}$ and $P_{x, y}$ are Laurent polynomials in $q^{\frac{1}{2}}$ with nonnegative integer coefficients for all $x, y \leq v$, by Propositions 5.1.6 and 5.1.7 we have

$$
\sum_{y \in A} L_{y} q^{-\frac{l(y)}{2}} P_{x, y} \leq \sum_{y \leq v} L_{y} q^{-\frac{l(y)}{2}} P_{x, y}=q^{-\frac{l}{2}} \sum_{\{S \in \mathcal{P}([l]): \pi(S)=x\}} q^{d(S, l)},
$$

where the $\leq$ is coefficientwise, and this implies (5.3).
Now let $y$ be a maximal element of $A \backslash\{v\}$ and $x \in[e, v]$. Comparing the
coefficients of $T_{x}$ on both sides of (5.3) we obtain that

$$
\begin{align*}
\sum_{\{S \in \mathcal{E}: \pi(S)=x\}} q^{d(S, l)} & =\sum_{z \in A} L_{z} q^{\frac{l(z, v)}{2}} P_{x, z}  \tag{5.8}\\
& = \begin{cases}L_{y} q^{\frac{l(y, v)}{2}}+P_{y, v}, & \text { if } x=y \\
P_{x, v}, & \text { if } y<x<v,\end{cases} \tag{5.9}
\end{align*}
$$

and (5.4) and (5.5) follow since $L_{y} \neq 0$ and $L_{y}(q)=L_{y}\left(q^{-1}\right)$. Conversely, let $y \in A \backslash\{v\}$ be such that (5.4) and (5.5) hold. Then, by (5.8),

$$
\operatorname{deg}\left(\sum_{z \in A} L_{z} q^{\frac{l(z, v)}{2}} P_{x, z}\right)<\frac{l(x, v)}{2}
$$

for all $y<x<v$. Since $L_{z}$ and $P_{x, z}$ are Laurent polynomials in $q^{\frac{1}{2}}$ with nonnegative coefficients for all $x, z \leq v$, this implies that $x \notin A$ for all $y<x<v$, so $y$ is maximal in $A \backslash\{v\}$.

Finally, if $y \in A \backslash\{v\}$ satisfies (5.4) and (5.5) then by (5.9) we have

$$
\sum_{\{S \in \mathcal{E}: \pi(S)=y\}} q^{d(S, l)}=L_{y} q^{\frac{l(y, v)}{2}}+P_{y, v}
$$

and (5.6) and (5.7) follow since $\operatorname{deg}\left(P_{y, v}\right)<\frac{l(y, v)}{2}$ and $L_{y} \in \mathbb{N}\left[q^{\frac{1}{2}}+q^{-\frac{1}{2}}\right]$.

Theorem 5.1.8 yields an inductive, entirely poset theoretic way of computing Kazhdan-Lusztig polynomials, which generalizes the one given in [28]. In fact, let $v \in W$ and assume that we have already computed the polynomials $P_{x, y}$ for all $x, y<v$. Take a regular sequence for $v$, and from it compute, for each $x \leq v$, using Propositions 5.1.6 and 5.1.7, the coefficient $\mathcal{P}_{x}$ of $T_{x}$ in

$$
q^{\frac{l(v)}{2}} \sum_{x \leq v} L_{x} C_{x}^{\prime}
$$

We apply Theorem 5.1.8 to the set $A:=\left\{x \in[e, v]: L_{x} \neq 0\right\}$. If $\operatorname{deg}\left(\mathcal{P}_{x}\right)<\frac{l(x, v)}{2}$ for all $x<v$, then by Theorem 5.1.8 there are no maximal elements in $A \backslash\{v\}$, namely $A=\{v\}$. Hence

$$
\sum_{x \leq v} L_{x} C_{x}^{\prime}=C_{v}^{\prime}
$$

and $\mathcal{P}_{x}=P_{x, v}$ for all $x \leq v$. Otherwise, let $y<v$ be a maximal element such
that $\operatorname{deg}\left(\mathcal{P}_{y}\right) \geq \frac{l(y, v)}{2}$. Then, by (5.6),

$$
q^{\frac{l(y, v)}{2}} L_{y}=U_{\frac{l(y, v)}{2}}\left(\mathcal{P}_{y}(q)\right)+D_{\frac{l(y, v)-1}{2}}\left(q^{l(y, v)} \mathcal{P}_{y}\left(\frac{1}{q}\right)\right) .
$$

where $U_{k}$ and $D_{k}$ are the linear operators satisfying:

$$
U_{k}\left(q^{i}\right)=\left\{\begin{array}{cl}
0, & \text { if } i<k, \\
q^{i}, & \text { if } i \geq k,
\end{array} \quad D_{k}\left(q^{i}\right)=\left\{\begin{array}{cl}
q^{i}, & \text { if } i \leq k, \\
0, & \text { if } i>k
\end{array}\right.\right.
$$

Since, by induction, we have already computed $P_{x, y}$ for all $x \in[e, v]$ we may compute the differences

$$
\begin{equation*}
\mathcal{P}_{x}^{\prime}=\mathcal{P}_{x}-q^{\frac{l(y, v)}{2}} L_{y} P_{x, y} \tag{5.10}
\end{equation*}
$$

for all $x \in[e, v]$. Clearly, $\mathcal{P}_{x}^{\prime}$ is the coefficient of $T_{x}$ in

$$
q^{\frac{l(v)}{2}} \sum_{x \in[e, v] \backslash\{y\}} L_{x} C_{x}^{\prime} .
$$

If $\operatorname{deg}\left(\mathcal{P}_{x}^{\prime}\right)<\frac{l(x, v)}{2}$ for all $x<v$ then Theorem 5.1.8 applied to $A \backslash\{y\}$ gives

$$
\sum_{x \in[e, v] \backslash\{y\}} L_{x} C_{x}^{\prime}=C_{v}^{\prime}
$$

and hence $\mathcal{P}_{x}^{\prime}=P_{x, v}$ for all $x \leq v$. Otherwise, let $y_{1}<v$ be a maximal element such that $\operatorname{deg}\left(\mathcal{P}_{y_{1}}^{\prime}\right) \geq \frac{l\left(y_{1}, v\right)}{2}$, and repeat the above procedure with $y_{1}$ in place of $y$ (note that $y_{1} \nsupseteq y$ by (5.10)). After at most $|[e, v]|-1$ steps this process will stop.

As an immediate consequence of Theorem 5.1.8 we obtain the following result which, in the case that the regular sequence comes from a reduced expression, is closely related to Theorem 4.12 of [28].

Corollary 5.1.9 Let $(W, S)$ be a nonnegative Coxeter system, $v \in W$, and $\left(M_{1}, \ldots, M_{l}\right)$ be a regular sequence for $v$. Then there exists $\mathcal{E} \subseteq \mathcal{P}([l])$ such that

$$
P_{u, v}(q)=\sum_{\{S \in \mathcal{E}: \pi(S)=u\}} q^{d(S, l)}
$$

for all $u<v$.
Proof. This follows immediately by taking $A=\{v\}$ in Theorem 5.1.8.

### 5.2 B-regular sequences

Our purpose in this section is to obtain a bijection between subsequences of certain regular sequences and certain paths in an appropriate directed graph. This bijection has several nice properties, and transforms the concepts and statistics used in the previous section into familiar ones on paths. The main results of this section are new even in the case that the regular sequence comes from a reduced expression.

Let $v \in W$ and $\mathcal{M}:=\left(M_{1}, \ldots, M_{l}\right)$ be a regular sequence for $v$. We say that $\mathcal{M}$ is $B$-regular if

$$
M_{i}(x) \neq M_{i+1} M_{i+2} \cdots M_{i+k} \cdots M_{i+2} M_{i+1}(x)
$$

for all $i \in[l], k \in[l-i]$, and for all $x \in[e, v]$ for which both sides are defined. Note that $\mathcal{M}$ is B-regular if and only if

$$
M_{i}(x) \neq M_{i-1} M_{i-2} \cdots M_{i-k} \cdots M_{i-2} M_{i-1}(x)
$$

for all $i \in[l], k \in[i-1]$, and for all $x \in[e, v]$ for which both sides are defined.
Let $v \in W$ and $\mathcal{M}:=\left(M_{1}, \ldots, M_{l}\right)$ be a $B$-regular sequence for $v$. The $B$-graph of $[e, v]$, with respect to $\mathcal{M}$, is the directed graph having $[e, v]$ as vertex set and where, for any $x, y \in[e, v], x \rightarrow y$ if and only if $l(x)<l(y)$ and there exists $i \in[l]$ such that

$$
y=M_{l} M_{l-1} \cdots M_{i+1} M_{i} M_{i+1} \cdots M_{l-1} M_{l}(x) .
$$

If $x \rightarrow y$, then, by the definition of $B$-regular, there is a unique $i \in[l]$ such that $y=M_{l} \cdots M_{i} \cdots M_{l}(x)$ (for if $M_{l} \cdots M_{i} \cdots M_{l}(x)=M_{l} \cdots M_{j} \cdots M_{l}(x)$ for some $1 \leq i<j \leq l$ then $M_{j}(\tilde{x})=M_{j-1} \cdots M_{i} \cdots M_{j-1}(\tilde{x})$ where $\tilde{x}:=$ $M_{j} \cdots M_{l}(x)$, which contradicts the fact that $\mathcal{M}$ is B-regular). We therefore define

$$
\lambda(x, y):=\lambda(y, x):=i .
$$

For example, one may easily check that the regular sequence in Figure 5.1 is actually $B$-regular. The corresponding $B$-graph is shown in Figure 5.2, where we have labeled all edges $x \rightarrow y$ with $\lambda(x, y)$, and we have kept all vertices in the same place for clarity.

Note that B-regular sequences always exist. In fact, given any reduced ex-


Figure 5.1: A B-regular sequence of special matchings.
pression $s_{1} s_{2} \cdots s_{n}$ of $v$, the sequences $\left(\lambda_{s_{n}}, \lambda_{s_{n-1}}, \ldots, \lambda_{s_{1}}\right)$ and $\left(\rho_{s_{1}}, \rho_{s_{2}}, \ldots, \rho_{s_{n}}\right)$ are B-regular, as it is easy to check. Therefore, the concept of a $B$-regular sequence is a generalization of that of a reduced decomposition.

One of the crucial properties of the $B$-graphs of lower intervals of Coxeter groups is that they are always directed subgraphs of the Bruhat graph. This hinges on the following result. Recall that we denote by $T$ the set of reflections of a Coxeter system $(W, S)$.

Theorem 5.2.1 Let $v \in W$, and $M$ be a special matching of $[e, v]$. Suppose $x, y \in[e, v]$ are such that $x^{-1} y \in T$. Then

$$
\begin{equation*}
M(x)^{-1} M(y) \quad \in \quad T \tag{5.11}
\end{equation*}
$$

Proof. We assume that $l(x)<l(y)$ and we proceed by induction on $l(x, y) \geq 1$.
If $l(x, y)=1$ then $x \triangleleft y$. If either $M(x) \triangleright x$ or $M(y) \triangleleft y$, then (5.11) follows immediately from the definition of a special matching. If $M(x) \triangleleft x \triangleleft y \triangleleft M(y)$


Figure 5.2: The B-graph corresponding to the B-regular sequence of Figure 5.1.
then, by Lemma 4.2.1, $M$ restricts to a special matching of $[M(x), M(y)]$. But it is well known (see, e.g., [11, Lemma 6.2]) that a Bruhat interval of rank 3 is isomorphic to a $k$-crown for some $k \geq 2$. On the other hand, it is easy to see that a $k$-crown has no special matchings if $k \geq 4$, while a 3 -crown has no special matching $M$ satisfying $M(\hat{0})<M(\hat{1})$. Hence $[M(x), M(y)]$ is a 2 -crown, so it is isomorphic to $S_{3}$, and it is known (see Proposition 3.3 of [32]) that this implies that $M(x)^{-1} M(y) \in T$.

Suppose now that $l(x, y) \geq 3$. From our hypotheses and (the proof of) Proposition 3.3 of [32], we have that necessarily there exist $a, b, c, d \in[x, y]$, all distinct, such that $l(x)<l(a)<l(c)<l(y), l(x)<l(b)<l(d)<l(y)$, and $\left\{x^{-1} a, a^{-1} c, c^{-1} y, x^{-1} b, b^{-1} d, d^{-1} y, a^{-1} d, b^{-1} c\right\} \subseteq T$. Therefore, from our induction hypothesis, we conclude that

$$
\left\{M(x)^{-1} M(a), M(a)^{-1} M(c), M(c)^{-1} M(y), M(x)^{-1} M(b),\right.
$$

$$
\begin{equation*}
\left.M(b)^{-1} M(d), M(d)^{-1} M(y), M(a)^{-1} M(d), M(b)^{-1} M(c)\right\} \subseteq T \tag{5.12}
\end{equation*}
$$

But $\left(M(x)^{-1} M(a)\right)\left(M(a)^{-1} M(c)\right)=\left(M(x)^{-1} M(b)\right)\left(M(b)^{-1} M(c)\right) \neq e$. Hence, by Proposition 4.1.1 (or by Lemma 3.1 of [32]),

$$
W_{x, a, b, c}:=\left\langle M(x)^{-1} M(a), M(a)^{-1} M(c), M(x)^{-1} M(b), M(b)^{-1} M(c)\right\rangle
$$

is a dihedral reflection subgroup of $W$. Similarly,

$$
W_{x, a, b, d}:=\left\langle M(x)^{-1} M(a), M(a)^{-1} M(d), M(x)^{-1} M(b), M(b)^{-1} M(d)\right\rangle
$$

and

$$
W_{b, c, d, y}:=\left\langle M(b)^{-1} M(c), M(c)^{-1} M(y), M(b)^{-1} M(d), M(d)^{-1} M(y)\right\rangle
$$

are dihedral reflection subgroups of $W$. But $W_{x, a, b, c} \cap W_{x, a, b, d} \supseteq\left\langle M(x)^{-1} M(a)\right.$, $\left.M(x)^{-1} M(b)\right\rangle$. Therefore, by Remark 3.2 of [32], there exists a dihedral reflection subgroup $W^{\prime}$ of $W$ such that $W^{\prime} \supseteq W_{x, a, b, c} \cup W_{x, a, b, d}$. Similarly, $W^{\prime} \cap W_{b, c, d, y} \supseteq\left\langle M(b)^{-1} M(c), M(b)^{-1} M(d)\right\rangle$, so there exists a dihedral reflection subgroup $W^{\prime \prime}$ of $W$ such that $W^{\prime \prime} \supseteq W^{\prime} \cup W_{b, c, d, y}$ (we could also have taken $W^{\prime}$ maximal so that $W^{\prime \prime}=W^{\prime}$ ). This implies that

$$
\{M(x), M(a), M(b), M(c), M(d), M(y)\} \subseteq M(x) W^{\prime \prime}
$$

By Theorem 1.4 of [32], there is an isomorphism of directed graphs $\phi$ from the graph induced on $M(x) W^{\prime \prime}$ by the Bruhat graph of $W$ to the Bruhat graph of $W^{\prime \prime}$ (considered as an abstract Coxeter system). Hence, by (5.12), in the Bruhat graph of $W^{\prime \prime}$ there are edges connecting $\phi(M(x))$ with $\phi(M(a))$, $\phi(M(a))$ with $\phi(M(c))$, and $\phi(M(c))$ with $\phi(M(y))$. But $W^{\prime \prime}$ is a dihedral Coxeter group, hence for any $u, w \in W^{\prime \prime}$ there is an edge in the Bruhat graph of $W^{\prime \prime}$ connecting $u$ with $w$ if and only if $l^{\prime \prime}(u, w) \equiv 1(\bmod 2)$, where $l^{\prime \prime}$ is the length function of $W^{\prime \prime}$ with respect to its set of canonical generators. Therefore $l^{\prime \prime}(\phi(M(x)), \phi(M(a))) \equiv l^{\prime \prime}(\phi(M(a)), \phi(M(c))) \equiv l^{\prime \prime}(\phi(M(c)), \phi(M(y))) \equiv 1$ $(\bmod 2)$, which implies that $l^{\prime \prime}(\phi(M(x)), \phi(M(y))) \equiv 1(\bmod 2)$, and hence that there is an edge, in the Bruhat graph of $W^{\prime \prime}$, connecting $\phi(M(x))$ with $\phi(M(y))$. But $\phi$ is an isomorphism of directed graphs, so there is an edge in the Bruhat graph of $W$ connecting $M(x)$ with $M(y)$, and (5.11) follows.

We can now prove that the $B$-graphs of lower intervals of a Coxeter system
are always directed subgraphs of the Bruhat graph.

Corollary 5.2.2 Let $v_{1}, \ldots, v_{r} \in W$ and $M_{i}$ be a special matching of $\left[e, v_{i}\right]$ for $i=1, \ldots, r$. Let $x \in W$ be such that $M_{r} M_{r-1} \cdots M_{2} M_{1} M_{2} \cdots M_{r-1} M_{r}(x)$ is defined. Then

$$
\begin{equation*}
x^{-1} M_{r} M_{r-1} \cdots M_{2} M_{1} M_{2} \cdots M_{r-1} M_{r}(x) \quad \in \quad T . \tag{5.13}
\end{equation*}
$$

Proof. We proceed by induction on $r \geq 1$, the result being clearly true if $r=1$. So assume that $r \geq 2$. From our hypothesis, it follows that the composition $M_{r-1} \cdots M_{2} M_{1} M_{2} \cdots M_{r-1}\left(M_{r}(x)\right)$ is defined. Hence, by our induction hypothesis, $M_{r}(x)^{-1} M_{r-1} \cdots M_{2} M_{1} M_{2} \cdots M_{r-1}\left(M_{r}(x)\right) \in T$. Therefore, by Theorem 5.2.1, $x^{-1} M_{r} M_{r-1} \cdots M_{2} M_{1} M_{2} \cdots M_{r-1} M_{r}(x) \in T$.

An important consequence of Corollary 5.2.2 is the following result, which in the case that the $B$-regular sequence $\left(M_{1}, \ldots, M_{l}\right)$ comes from a reduced decomposition is a consequence of the Exchange Property.

Proposition 5.2.3 Let $v \in W,\left(M_{1}, \ldots, M_{l}\right)$ be a $B$-regular sequence for $v$, and $y \in[e, v], j \in[l]$ be such that $M_{j}(y)$ is defined. Then the following are equivalent:
i) $M_{j}(y) \triangleright y$;
ii) $M_{l} \cdots M_{j}(y)>M_{l} \cdots M_{j+1}(y)$.

Proof. Assume first that i) holds. We will prove, by induction on $k$, that

$$
\begin{equation*}
M_{j+k} \cdots M_{j}(y)>M_{j+k} \cdots M_{j+1}(y) \tag{5.14}
\end{equation*}
$$

for $k=0, \ldots, l-j$. If $k=0$ then (5.14) is true by our hypothesis i). So let $k \geq 1$ and assume, by induction, that

$$
\begin{equation*}
a:=M_{j+k-1} \cdots M_{j}(y)>M_{j+k-1} \cdots M_{j+1}(y):=b \tag{5.15}
\end{equation*}
$$

Note that

$$
M_{j+k}(a)=M_{j+k} \cdots M_{j+1} M_{j} M_{j+1} \cdots M_{j+k}\left(M_{j+k}(b)\right)
$$

Therefore, by Corollary 5.2.2, $M_{j+k}(a)$ and $M_{j+k}(b)$ are comparable in the

Bruhat order. Hence, to prove (5.14), it is enough to show that

$$
\begin{equation*}
l\left(M_{j+k}(a)\right) \geq l\left(M_{j+k}(b)\right) . \tag{5.16}
\end{equation*}
$$

Suppose, by contradiction, that

$$
\begin{equation*}
l\left(M_{j+k}(a)\right)<l\left(M_{j+k}(b)\right) . \tag{5.17}
\end{equation*}
$$

From (5.15) we have that $l(a)>l(b)$. This, together with (5.17), forces that $b \triangleleft a$ and this implies that $M_{j+k}(b)=a$, since $M_{j+k}$ is a special matching. Therefore

$$
M_{j+k}(b)=M_{j+k-1} \cdots M_{j+1} M_{j} M_{j+1} \cdots M_{j+k-1}(b)
$$

and this contradicts the hypothesis that $\left(M_{1}, \ldots, M_{l}\right)$ is a $B$-regular sequence. This proves (5.16) and hence (5.14) and concludes the induction step.

Assume now that i) does not hold, i.e. $M_{j}(y) \triangleleft y$. Then $M_{j}\left(M_{j}(y)\right) \triangleright M_{j}(y)$. Hence, by what we have just proved

$$
M_{l} \cdots M_{j} M_{j}(y)>M_{l} \cdots M_{j+1} M_{j}(y)
$$

so ii) does not hold.
Note that the above proposition does not hold if $\left(M_{1}, \ldots, M_{l}\right)$ is regular but not $B$-regular. For example, let $W=\mathfrak{S}(5), v=32154,\left(M_{1}, \ldots, M_{4}\right)=$ $\left(\rho_{2}, \rho_{1}, \rho_{4}, \lambda_{1}\right), y=e$, and $j=2$. Then $\left(M_{1}, \ldots, M_{4}\right)$ is a regular sequence for $v$ and $M_{2}(e) \triangleright e$ but $M_{4} M_{3} M_{2}(e)=12354 \not 221354=M_{4} M_{3}(e)$.

We can now prove the main result of this section, which gives a bijection between subsequences of a $B$-regular sequence and certain paths in the $B$-graph of $[e, v]$. The result is new even in the case that the $B$-regular sequence comes from a reduced decomposition. Recall the definition of $\pi, d_{1}(S, l)$ and $d_{2}(S)$ from Section 5.1.

Theorem 5.2.4 Let $v \in W$ and $\left(M_{1}, \ldots, M_{l}\right)$ be a $B$-regular sequence for $v$. Then there is a bijection between subsets $S$ of $[l]$ and (undirected) paths $\Delta=\left(x_{0}, x_{1}, \ldots, x_{s}\right)$ in the B-graph of $[e, v]$ such that $x_{0}=v$ and $\lambda\left(x_{0}, x_{1}\right)<$ $\lambda\left(x_{1}, x_{2}\right)<\cdots<\lambda\left(x_{s-1}, x_{s}\right)$. Furthermore:
i) $l(\Delta)=l-|S|$;
ii) $x_{s}=\pi(S)$;
iii) $d_{1}(S, l)=\left|\left\{i \in[s]: x_{i-1}<x_{i}\right\}\right|$;
iv) $d_{2}(S)=\frac{1}{2}\left(l-l\left(x_{s}\right)-l(\Delta)\right)$.

Proof. For $S=\left\{i_{1}, \ldots, i_{k}\right\}_{<} \subseteq[l]$ let $\left\{j_{1}, \ldots, j_{s}\right\}_{<}:=[l] \backslash S$ and

$$
x_{i}:=R_{j_{i}} \cdots R_{j_{2}} R_{j_{1}}(v)
$$

for $i=0, \ldots, s$, where $R_{i}:=M_{l} \cdots M_{i} \cdots M_{l}$ for $i \in[l]$. Then $x_{i}=R_{j_{i}}\left(x_{i-1}\right)$ and hence $\lambda\left(x_{i-1}, x_{i}\right)=j_{i}$ for $i \in[s]$. Clearly $s=l-k$ and

$$
\begin{aligned}
x_{i} & =R_{j_{i}} \cdots R_{j_{2}} R_{j_{1}} M_{l} \cdots M_{1}(e) \\
& =M_{l} \cdots \widehat{M}_{j_{i}} \cdots \widehat{M}_{j_{2}} \cdots \widehat{M}_{j_{1}} \cdots M_{1}(e) \\
& =M_{l} \cdots M_{j_{i}+1}(y),
\end{aligned}
$$

where $y=\pi\left(S \cap\left[j_{i}-1\right]\right)$, for each $i \in[s]$. Hence $x_{s}=\pi(S)$ and, for $i \in[s]$, $x_{i-1}<x_{i}$ if and only if

$$
R_{j_{i}}\left(x_{i}\right)=M_{l} \cdots M_{j_{i}}(y)<M_{l} \cdots M_{j_{i}+1}(y)=x_{i}
$$

which, by Proposition 5.2.3, happens if and only if

$$
M_{j_{i}}(y) \triangleleft y
$$

namely if and only if $\varepsilon_{j_{i}}(S)=1$. This proves iii).
Finally, by ii),

$$
\begin{aligned}
l\left(x_{s}\right) & =k-2\left|\left\{a \in[k]: \quad M_{i_{a}} M_{i_{a-1}} \cdots M_{i_{1}}(e) \triangleleft M_{i_{a-1}} \cdots M_{i_{1}}(e)\right\}\right| \\
& =k-2 \sum_{a \in[k]} \varepsilon_{i_{a}}(S) \\
& =k-2 d_{2}(S) .
\end{aligned}
$$

It is clear that this map $S \mapsto\left(x_{0}, x_{1}, \ldots, x_{s}\right)$ is a bijection.
Combining Theorems 5.2.4 and 5.1.4 we obtain the following result.

Corollary 5.2.5 Let $v \in W$, and $\left(M_{1}, \ldots, M_{l}\right)$ be a $B$-regular sequence for $v$. Then, for all $u \leq v$,

$$
\widetilde{R}_{u, v}(q)=\sum_{\Delta} q^{l(\Delta)}
$$

where $\Delta$ runs over all the directed paths $u=x_{s} \rightarrow \ldots \rightarrow x_{2} \rightarrow x_{1} \rightarrow x_{0}=v$ in the $B$-graph of $[e, v]$ such that $\lambda\left(x_{0}, x_{1}\right)<\lambda\left(x_{1}, x_{2}\right)<\ldots<\lambda\left(x_{s-1}, x_{s}\right)$.

In the case that the $B$-regular sequence comes from a reduced expression Corollary 5.2.5 is closely related to (but not implied by) Corollary 3.4 of [33].

We illustrate Corollary 5.2 .5 with an example. Consider the $B$-regular sequence $\left(M_{1}, \ldots, M_{5}\right)$ illustrated in Figure 5.1. Then by Corollary 5.2 .5 we can "read off" from the corresponding $B$-graph (Figure 5.2) that, for example,

$$
\widetilde{R}_{e, v}(q)=q^{5}+2 q^{3}+q,
$$

corresponding to the directed paths from $e$ to $v$ having sequences of labels $(5,4,3,2,1),(5,3,2),(4,3,1)$ and (3).

Combining Theorem 5.2.4 with Corollary 5.1.9 we obtain the following result, which appears to be new even in the case that the $B$-regular sequence comes from a reduced decomposition.

Corollary 5.2.6 Let $(W, S)$ be a nonnegative Coxeter system, $v \in W$, and $\left(M_{1}, \ldots, M_{l}\right)$ be a B-regular sequence for $v$. Then there is a subset $\mathcal{E}$ of the set of (undirected) paths $\Delta=\left(x_{0}, x_{1}, \ldots, x_{l(\Delta)}\right)$ in the $B$-graph of $[e, v]$ satisfying $x_{0}=v$ and $\lambda\left(x_{0}, x_{1}\right)<\lambda\left(x_{1}, x_{2}\right)<\cdots<\lambda\left(x_{l(\Delta)-1}, x_{l(\Delta)}\right)$, such that

$$
P_{u, v}(q)=\sum_{\left\{\Delta \in \mathcal{E}: x_{l(\Delta)}=u\right\}} q^{\frac{1}{2}(l(u, v)+l(\Delta)-2 d(\Delta))}
$$

for all $u \leq v$, where $d(\Delta)=\left|\left\{i \in[l(\Delta)]: x_{i-1}>x_{i}\right\}\right|$.

Note that the subset $\mathcal{E}$ can be determined using the algorithm in Section 5.1 and Theorem 5.1.8.

## $5.3 R$-regular sequences

In this section we generalize to a combinatorially invariant setting what is probably the most explicit non-recursive formula known for Kazhdan-Lusztig polynomials which holds in complete generality, namely Theorem 7.3 of [14]. In the following two subsections we introduce the preliminary results that will be needed in the third subsection.

### 5.3.1 Reflection orderings

Let $(G, S)$ be any Coxeter system. Following [33] we say that a total ordering $\prec$ of the set of reflections $T$ of $G$ is a reflection ordering if, for any dihedral reflection subgroup ( $G^{\prime},\{a, b\}$ ), where $a, b$ are the canonical generators of $G^{\prime}$, we have that either $a \prec a b a \prec a b a b a \prec \cdots \prec b a b a b \prec b a b \prec b$ or $b \prec b a b \prec \cdots \prec a b a \prec a$. It can be proved that such orderings always exist (see [33]).

Let $\prec$ be a reflection ordering, and $s \in S$. We define a total ordering $\prec^{s}$ on $T$ as follows. For $t_{1}, t_{2} \in T$ we set $t_{1} \prec^{s} t_{2}$ if and only if either one of the following conditions apply:

1. $t_{1}, t_{2} \prec s$ and $t_{1} \prec t_{2}$;
2. $t_{1}, t_{2} \succ s$ and $s t_{1} s \prec s t_{2} s ;$
3. $t_{1} \prec s \prec t_{2}$;
4. $t_{2}=s$.

Similarly, we define $\prec_{s}$ by letting $t_{1} \prec_{s} t_{2}$ if and only if either one of the following conditions is satisfied:

1. $t_{1}, t_{2} \prec s$ and $s t_{1} s \prec s t_{2} s ;$
2. $t_{1}, t_{2} \succ s$ and $t_{1} \prec t_{2}$;
3. $t_{1} \prec s \prec t_{2}$;
4. $t_{1}=s$.

It can be proved (see Poposition 2.5 of [33]) that these orders are well-defined and that they are still reflection orderings. Note that

$$
\begin{equation*}
\left(\prec_{s}\right)^{s}=\prec^{s} . \tag{5.18}
\end{equation*}
$$

### 5.3.2 Chains and lattice paths

For $j \in \mathbb{Q}$ we define an operator $L_{j}: \mathbb{C}[q] \rightarrow \mathbb{C}[q]$ by letting

$$
L_{j}\left(\sum_{i \geq 0} a_{i} q^{i}\right):=\sum_{0 \leq i \leq j} a_{i} q^{i}
$$

Following [14], given a chain $\mathcal{C}=x_{0}<x_{1}<\cdots<x_{n}$ in $W$ of length $l(\mathcal{C}):=n$, we define

$$
R_{x_{0}, \ldots, x_{n}}(q):=R_{x_{0}, x_{1}}(q) L_{\frac{d-1}{2}}\left(R_{x_{1}, \ldots, x_{n}}(q)\right),
$$

where $d:=l\left(x_{1}, x_{n}\right)$, if $n \geq 2$ and

$$
R_{x_{0}, \ldots, x_{n}}(q):=R_{x_{0}, x_{n}}(q),
$$

if $n=1$, where the right-hand side is the usual $R$-polynomial. The polynomial $R_{x_{0}, \ldots, x_{n}}(q)$ is called the $R$-polynomial of the chain $x_{0}<x_{1}<\cdots<x_{n}$. The following result appeared in [14, Theorem 4.1] and is a non-recursive formula for Kazhdan-Lusztig polynomials in terms of the $R$-polynomials of a chain.

Theorem 5.3.1 Let $W$ be a Coxeter group and $u, v \in W, u \leq v$. Then

$$
P_{u, v}(q)-q^{l(u, v)} P_{u, v}\left(q^{-1}\right)=\sum_{\mathcal{C} \in C(u, v)}(-1)^{l(\mathcal{C})} R_{\mathcal{C}}(q)
$$

where $C(u, v)$ is the set of all chains from $u$ to $v$.

Recall that a composition of $n \in \mathbb{P}$ is a sequence $\left(\alpha_{1}, \ldots, \alpha_{s}\right)$ (for some $s \in$ $\mathbb{P}$ ) of positive integers such that $\alpha_{1}+\ldots+\alpha_{s}=n$. When writing compositions we will sometimes omit to write the parentheses (i.e., we will write $\alpha_{1}, \ldots, \alpha_{s}$ instead of $\left.\left(\alpha_{1}, \ldots, \alpha_{s}\right)\right)$. For $n \in \mathbb{P}$ we let $C_{n}$ be the set of all compositions of $n$ and $C:=\bigcup_{n \geq 1} C_{n}$. Given $\beta \in C$ we denote by $l(\beta)$ the number of parts of $\beta$, by $\beta_{i}$, for $i=1, \ldots l(\beta)$, the $i$-th part of $\beta$ (so that $\beta=\left(\beta_{1}, \beta_{2}, \ldots, \beta_{l(\beta)}\right)$ ), and we let $|\beta|:=\sum_{i=1}^{l(\beta)} \beta_{i}, \bar{\beta}:=\left(\beta_{2}, \beta_{3}, \ldots, \beta_{l(\beta)}\right)$ (if $l(\beta) \geq 2), \beta^{*}:=\left(\beta_{l(\beta)}, \ldots, \beta_{2}, \beta_{1}\right), T(\beta):=\left\{\beta_{r}, \beta_{r}+\beta_{r-1}, \ldots, \beta_{r}+\ldots+\beta_{2}\right\}$ where $r:=l(\beta)$. Given $\left(\alpha_{1}, \ldots, \alpha_{s}\right),\left(\beta_{1}, \ldots \beta_{t}\right) \in C_{n}$ we say that $\left(\alpha_{1}, \ldots, \alpha_{s}\right)$ refines $\left(\beta_{1}, \ldots \beta_{t}\right)$ if there exist $1 \leq i_{1}<i_{2}<\cdots<i_{t-1} \leq s$ such that $\sum_{j=i_{k-1}+1}^{i_{k}} \alpha_{j}=\beta_{k}$ for $k=1, \ldots, t$ (where $i_{0}:=0, i_{t}:=s$ ). We then write $\left(\alpha_{1}, \ldots, \alpha_{s}\right) \leq_{c}\left(\beta_{1}, \ldots \beta_{t}\right)$. It is easy to see that the map $\beta \mapsto T(\beta)$ is an isomorphism from $\left(C_{n}, \leq_{c}\right)$ to the Boolean algebra of subsets of $[n-1]$ ordered by reverse inclusion.

Let $n \in \mathbb{N}$. By a lattice path of length $n$ we mean a function $\Gamma:[0, n] \rightarrow \mathbb{Z}$ such that $\Gamma(0)=0$ and

$$
|\Gamma(i)-\Gamma(i-1)|=1
$$

for all $i \in[n]$. Given such a lattice path $\Gamma$ we let

$$
\begin{gathered}
N(\Gamma):=\{i \in[n-1]: \Gamma(i)<0\} \\
d_{+}(\Gamma):=|\{i \in[0, n-1]: \Gamma(i+1)-\Gamma(i)=1\}|,
\end{gathered}
$$

$l(\Gamma):=n$, and $\Gamma_{\geq 0}:=l(\Gamma)-1-|N(\Gamma)|$. We call $N(\Gamma)$ the negative set of $\Gamma$, and $l(\Gamma)$ the length of $\Gamma$. Note that $n \notin N(\Gamma)$ and that

$$
\begin{equation*}
d_{+}(\Gamma)=\frac{\Gamma(n)+n}{2} . \tag{5.19}
\end{equation*}
$$

Let $\mathcal{L}(n)$ denote the set of all lattice paths of length $n$. Given $S \subseteq[n-1]$ we let

$$
H(S, n):=\{\Gamma \in \mathcal{L}(n): N(\Gamma) \supseteq S\}
$$

and

$$
E(S, n):=\{\Gamma \in \mathcal{L}(n): N(\Gamma)=S\} .
$$

For $\alpha \in C_{n}$ we define two polynomials $\Psi_{\alpha}(q), \Upsilon_{\alpha}(q) \in \mathbb{Z}[q]$ by letting

$$
\begin{equation*}
\Psi_{\alpha}(q):=(-1)^{n} \sum_{\Gamma \in H(T(\alpha), n)}(-q)^{d_{+}(\Gamma)}, \tag{5.20}
\end{equation*}
$$

and

$$
\Upsilon_{\alpha}(q):=(-1)^{n-l(\alpha)} \sum_{\Gamma \in E(T(\alpha), n)}(-q)^{d_{+}(\Gamma)} .
$$

Note that the definitions imply that

$$
\Psi_{\beta}(q)=\sum_{\alpha \leq_{c} \beta}(-1)^{l(\alpha)} \Upsilon_{\alpha}(q) .
$$

Hence, by the Principle of Inclusion-Exclusion,

$$
\begin{equation*}
\Upsilon_{\beta}(q)=\sum_{\alpha \leq{ }_{c} \beta}(-1)^{l(\alpha)} \Psi_{\alpha}(q) . \tag{5.21}
\end{equation*}
$$

The next result gives the $R$-polynomial of a chain in terms of the usual $\tilde{R}$ polynomials and its proof can be found in [14, Proposition 7.1].

Proposition 5.3.2 Let $x_{0}<x_{1}<\ldots<x_{n}$ be a chain in $W$. Then

$$
\begin{equation*}
R_{x_{0}, \ldots, x_{n}}(q)=\sum_{\alpha \in \mathbb{P}^{n}} q^{\frac{l\left(x_{0}, x_{n}\right)-|\alpha|}{2}} \Psi_{\alpha}(q) \prod_{r=1}^{n}\left[q^{\alpha_{r}}\right]\left(\widetilde{R}_{x_{r-1}, x_{r}}\right) . \tag{5.22}
\end{equation*}
$$

### 5.3.3 Poset theoretic formula

Let $v \in W$, and $\mathcal{M}:=\left(M_{1}, \ldots, M_{l}\right)$ be a regular sequence for $v$. We denote by $P_{\mathcal{M}}$ the set of palindromes in the alphabet $\left\{M_{1}, \ldots, M_{l}\right\}$, i.e. words of the form $M_{i_{1}} \cdots M_{i_{k-1}} M_{i_{k}} M_{i_{k-1}} \cdots M_{i_{1}}$ with $i_{1}, \ldots, i_{k} \in[l]$. We say that $\mathcal{M}$ is a reflection regular sequence, or simply an $R$-regular sequence, for $v$, if:
i) for $p_{1}, p_{2} \in P_{\mathcal{M}}$, if $p_{1}\left(u_{0}\right)=p_{2}\left(u_{0}\right)$ for some $u_{0} \in[e, v]$ then $p_{1}(u)=p_{2}(u)$ for all $u \in[e, v]$ for which both sides are defined;
ii) for $p_{1}, p_{2}, \ldots, p_{n} \in P_{\mathcal{M}}$, if $p_{i}$ and $p_{i+1}$ coincide on a point, for each $i=$ $1, \ldots, n-1$, then $p_{1}$ and $p_{n}$ coincide where they are both defined;
iii) $\mathcal{M}$ admits a reflection labeling.

We now define reflection labelings. Define an equivalence relation $\sim$ on $P_{\mathcal{M}}$ by letting $p_{1} \sim p_{2}$ if there exists $u_{0} \in[e, v]$ such that $p_{1}\left(u_{0}\right)=p_{2}\left(u_{0}\right)$ and taking the transitive closure. Note that this is stronger than requiring that $p_{1}(u)=p_{2}(u)$ for all $u \in[e, v]$ for which both sides are defined. We denote by $R_{\mathcal{M}}:=P_{\mathcal{M}} / \sim$ the quotient set. If $p \in P_{\mathcal{M}}$ we let $\bar{p}$ be the corresponding class in $R_{\mathcal{M}}$. Note that, for each $i, j \in[l], \overline{M_{i}}=\overline{M_{j}}$ if and only if $M_{i}(e)=M_{j}(e)$. Therefore, by Lemma 5.1.1, we may identify $\left\{\overline{M_{i}}: i \in[l]\right\}$ with the set of atoms of $[e, v]$. We say that an element $r \in R_{\mathcal{M}}$ is defined on some $u \in[e, v]$ if $p(u)$ is defined for some $p \in r$. In this case we write $r(u):=p(u)$. Now let $\left(W^{\prime}, S^{\prime}\right)$ be another Coxeter system and $T^{\prime}$ be its set of reflections. A reflection labeling of $R_{\mathcal{M}}$ in ( $W^{\prime}, S^{\prime}$ ) is a map $L: R_{\mathcal{M}} \rightarrow T^{\prime}$ such that:
a) $\left\{L\left(\overline{M_{i}}\right): i \in[l]\right\}=S^{\prime}$;
b) $L\left(\overline{M_{i_{1}} \cdots M_{i_{k}} \cdots M_{i_{1}}}\right)=L\left(\overline{M_{i_{1}}}\right) \cdots L\left(\overline{M_{i_{k}}}\right) \cdots L\left(\overline{M_{i_{1}}}\right)$ for all $i_{1}, \ldots, i_{k} \in$ [l];
c) If $r_{1}, r_{2} \in R_{\mathcal{M}}, r_{1} \neq r_{2}$, are both defined on some $u \in[e, v]$ then $L\left(r_{1}\right) \neq$ $L\left(r_{2}\right)$.

In particular $\left|S^{\prime}\right|$ equals the number of atoms of $[e, v]$.

It is not hard to see that $R$-regular sequences always exist. In fact, if $v=$ $s_{1} \cdots s_{l}$ is a reduced expression for $v$ then $\mathcal{M}:=\left(\rho_{1}, \ldots, \rho_{l}\right)$ is clearly a regular sequence for $v$ satisfying i) and ii). If we denote by $W^{\prime}$ the parabolic subgroup of $W$ generated by $\left\{s_{i}: i \in[l]\right\}$ and by $T^{\prime}$ its set of reflections, then the map $L$ : $P_{\mathcal{M}} \longrightarrow T^{\prime}$ defined by $\rho_{i_{1}} \cdots \rho_{i_{k}} \cdots \rho_{i_{1}} \mapsto s_{i_{1}} \cdots s_{i_{k}} \cdots s_{i_{1}}$ clearly factors through $R_{\mathcal{M}}$ to a reflection labeling. Similarly for $\left(\lambda_{l}, \ldots, \lambda_{1}\right)$. Thus, the concept of an $R$-regular sequence is a generalization of that of a reduced decomposition.

Although this is not obvious from the definition, an $R$-regular sequence is also $B$-regular.

Proposition 5.3.3 Let $v \in W$ and $\mathcal{M}$ be an $R$-regular sequence for $v$. Then $\mathcal{M}$ is $B$-regular.

Proof. Let $\mathcal{M}:=\left(M_{1}, \ldots, M_{l}\right)$ and fix $i \in[l]$. We will show that

$$
M_{i}(x) \neq M_{i-1} \cdots M_{i-k} \cdots M_{i-1}(x)
$$

for all $k \in[i-1]$ and all $x \in[e, v]$ for which both sides are defined, and the result will follow from the remarks following the definition of a $B$-regular sequence in Section 5.2.

Suppose, by contradiction, that there are $x \in[e, v]$ and $k \in[i-1]$ such that $M_{i}(x)=M_{i-1} \cdots M_{i-k} \cdots M_{i-1}(x)$. Since $\mathcal{M}$ is $R$-regular this implies, by condition i), that $M_{i}(y)=M_{i-1} \cdots M_{i-k} \cdots M_{i-1}(y)$ for all $y \in[e, v]$ for which both sides are defined. Let $\left(v_{0}, \ldots, v_{l}\right)$ be the regular chain associated to $\mathcal{M}$. Then, in particular,

$$
v_{i}=M_{i}\left(v_{i-1}\right)=M_{i-1} \cdots M_{i-k} \cdots M_{i-1}\left(v_{i-1}\right)=M_{i-1} \cdots M_{i-k+1}\left(v_{i-k-1}\right) .
$$

Therefore

$$
i=l\left(v_{i}\right)=l\left(M_{i-1} \cdots M_{i-k+1}\left(v_{i-k-1}\right)\right) \leq l\left(v_{i-k-1}\right)+k-1=i-2,
$$

which is a contradiction.
Note that the converse of the above proposition is not true. For example, let $W=\mathfrak{S}(4)$ and $v=3421$. Then it is easy to check that $\mathcal{M}:=\left(\rho_{2}, \rho_{3}, \rho_{2}, \lambda_{1}, \lambda_{2}\right)$ is a $B$-regular sequence for $v$. However, $\mathcal{M}$ is not $R$-regular since $\rho_{2}(e)=\lambda_{2}(e)$ but $\rho_{2}(1243) \neq \lambda_{2}(1243)$, so condition i) does not hold.

If $L: R_{\mathcal{M}} \rightarrow T$ is a reflection labeling and $\prec$ is a reflection ordering on $T$ we write, for brevity, $\prec^{i}:=\prec^{L\left(\overline{M_{i}}\right)}$ and $\prec_{i}:=\prec_{L\left(\overline{M_{i}}\right)}$.

Let $w \in W, \mathcal{M}$ an $R$-regular sequence for $v$, and $L: R_{\mathcal{M}} \rightarrow T^{\prime}$ be a reflection labeling. We define a labeled directed graph, that we call the $R$-graph of $[e, v]$ with respect to $\mathcal{M}$, as follows. The $R$-graph has $[e, v]$ as vertex set and, for any $x, y \in[e, v], x \xrightarrow{r} y$ if and only if $l(y)>l(x)$ and $y=r(x)$, for some $r \in R_{\mathcal{M}}$. Note that, by Corollary 5.2.2, the $R$-graph is a directed subgraph of the Bruhat graph.

If $\Delta=\left(x_{0} \xrightarrow{r_{1}} x_{1} \xrightarrow{r_{2}} \cdots \xrightarrow{r_{k}} x_{k}\right)$ is a path in the $R$-graph we write $E(\Delta):=\left\{r_{1}, \ldots, r_{k}\right\}$ and if $\prec$ is a reflection ordering on $T^{\prime}$ we let

$$
\begin{equation*}
D(\Delta, L, \prec):=\left\{i \in[k-1]: L\left(r_{i}\right) \succ L\left(r_{i+1}\right)\right\} . \tag{5.23}
\end{equation*}
$$

Finally, we define an element $R_{\prec}$ in the incidence algebra of $[e, v]$ by letting

$$
R_{\prec}(x, y):=\sum_{\{\Delta \in B(x, y): D(\Delta, L, \prec)=\emptyset\}} q^{l(\Delta)}
$$

where $B(x, y)$ denotes the set of all paths in the $R$-graph from $x$ to $y$.
We can now prove the first main result of this section. It is a "global version" of Corollary 5.2.5 and generalizes Corollary 3.4 of [33].

Theorem 5.3.4 Let $v \in W, \mathcal{M}=\left(M_{1}, \ldots, M_{l}\right)$ be an $R$-regular sequence for $v, L: R_{\mathcal{M}} \rightarrow T$ be a reflection labeling and $\prec a$ reflection ordering on $T$. Then

$$
\widetilde{R}_{x, y}(q)=R_{\prec}(x, y)
$$

for all $x \leq y \leq v$.
Proof. We proceed by induction on $l(y)$ the statement being trivial for $l(y)=0$.
Assume that $l(y)>0$. By Lemma 5.1.1 there is $i \in[l]$ such that $M_{i}(y) \triangleleft y$. Let, for brevity, $M:=M_{i}$. For all $x^{\prime}, y^{\prime} \leq y$ we let

$$
f_{\prec}\left(x^{\prime}, y^{\prime}\right):=\sum_{\Delta \in B_{i}\left(x^{\prime}, y^{\prime}\right)} q^{l(\Delta)}
$$

and

$$
g_{\prec}\left(x^{\prime}, y^{\prime}\right):=\sum_{\Delta \in B_{i}^{\prime}\left(x^{\prime}, y^{\prime}\right)} q^{l(\Delta)},
$$

where $B_{i}\left(x^{\prime}, y^{\prime}\right):=\left\{\Delta \in B\left(x^{\prime}, y^{\prime}\right): L(\bar{M}) \preceq L(E(\Delta))\right.$ and $\left.D(\Delta, L, \prec)=\emptyset\right\}$ and $B_{i}^{\prime}\left(x^{\prime}, y^{\prime}\right):=\left\{\Delta \in B\left(x^{\prime}, y^{\prime}\right): L(\bar{M}) \preceq L(E(\Delta))\right.$ and $\left.D\left(\Delta, L, \prec^{\prime}\right)=\emptyset\right\}$, where $\prec^{\prime}:=\prec^{L(\bar{M})}$.

We claim that

$$
f_{\prec}(x, y)= \begin{cases}g_{\prec}(M x, M y), & \text { if } M x \triangleleft x  \tag{5.24}\\ g_{\prec}(M x, M y)+q g_{\prec}(x, M y), & \text { otherwise }\end{cases}
$$

and

$$
g_{\prec}(x, y)= \begin{cases}f_{\prec}(M x, M y)+q\left(g_{\prec}(x, M y)-f_{\prec}(x, M y)\right), & \text { if } M x \triangleleft x  \tag{5.25}\\ f_{\prec}(M x, M y)+q g_{\prec}(x, M y), & \text { otherwise }\end{cases}
$$

where, for all $x \leq y$, we write $M x$ instead of $M(x)$.
We prove only the cases $M x \triangleleft x$ in equations (5.24) and (5.25), the cases $M x \triangleright x$ being similar. So suppose $M x \triangleleft x$.

Let $\Delta=\left(x_{0} \xrightarrow{r_{1}} x_{1} \xrightarrow{r_{2}} \cdots \xrightarrow{r_{k}} x_{k}\right) \in B_{i}(x, y)$. Then $\bar{M} \notin E(\Delta)$ and the path $\Delta^{\prime}=\left(M x_{0} \xrightarrow{r_{1}^{M}} M x_{1} \xrightarrow{r_{2}^{M}} \cdots \xrightarrow{r_{k}^{M}} M x_{k}\right)$, where $r^{M}:=\overline{M p M}$ for some (any) $p \in r$ belongs to $B_{i}^{\prime}(M x, M y)$. Conversely, every path in $B_{i}^{\prime}(M x, M y)$ arises in this way as it cannot have labels $\bar{M}$ since $M(y) \triangleleft y$. This proves the case $M x \triangleleft x$ of (5.24).

Now let $\Delta=\left(x_{0} \xrightarrow{r_{1}} x_{1} \xrightarrow{r_{2}} \cdots \xrightarrow{r_{k}} x_{k}\right) \in B_{i}^{\prime}(x, y)$. If $\bar{M} \in E(\Delta)$ then necessarily $\bar{M}=r_{k}$. Hence $\Delta^{\prime}=\left(x_{0} \xrightarrow{r_{1}} x_{1} \xrightarrow{r_{2}} \cdots \xrightarrow{r_{k-1}} x_{k-1}\right) \in B_{i}^{\prime}(x, M y)$. Furthermore, every path in $B_{i}^{\prime}(x, M y)$ cannot have $\bar{M}$ as a label and hence arises in this way from a path $\Delta \in B_{i}^{\prime}(x, y)$ such that $\bar{M} \in E(\Delta)$. So

$$
\begin{equation*}
\sum_{\left\{\Delta \in B_{i}^{\prime}(x, y): \bar{M} \in E(\Delta)\right\}} q^{l(\Delta)}=q g_{\prec}(x, M y) . \tag{5.26}
\end{equation*}
$$

If $\bar{M} \notin E(\Delta)$ then $\Delta^{\prime}=\left(M x_{0} \xrightarrow{r_{1}^{M}} M x_{1} \xrightarrow{r_{2}^{M}} \cdots \xrightarrow{r_{k}^{M}} M x_{k}\right) \in B_{i}(M x, M y)$. Moreover, any path $\Delta^{\prime} \in B_{i}(M x, M y)$ with $\bar{M} \notin E\left(\Delta^{\prime}\right)$ arises in this way. Hence

$$
\begin{equation*}
\sum_{\left\{\Delta \in B_{i}^{\prime}(x, y): \bar{M} \notin E(\Delta)\right\}} q^{l(\Delta)}=f_{\prec}(M x, M y)-\sum_{\left\{\Delta \in B_{i}(M x, M y): \bar{M} \in E(\Delta)\right\}} q^{l(\Delta)} \tag{5.27}
\end{equation*}
$$

Now let $\Delta=\left(x_{0} \xrightarrow{r_{1}} x_{1} \xrightarrow{r_{2}} \cdots \xrightarrow{r_{k}} x_{k}\right) \in B_{i}(M x, M y)$ be such that $\bar{M} \in$ $E(\Delta)$. Then necessarily $\bar{M}=r_{1}$ and hence $\Delta^{\prime \prime}=\left(x_{1} \xrightarrow{r_{2}} x_{2} \xrightarrow{r_{3}} \cdots \xrightarrow{r_{k}} x_{k}\right) \in$ $B_{i}(x, M y)$. Furthermore, every path in $B_{i}(x, M y)$ cannot have $\bar{M}$ as a label and hence arises in this way from a path $\Delta \in B_{i}(M x, M y)$ such that $\bar{M} \in E(\Delta)$.

Therefore

$$
\sum_{\left\{\Delta \in B_{i}(M x, M y): \bar{M} \in E(\Delta)\right\}} q^{l(\Delta)}=q f_{\prec}(x, M y)
$$

and this, together with (5.26) and (5.27), concludes the proof of the case $M x \triangleleft x$ of (5.25).

Now let

$$
h_{\prec}(x, y):=\sum_{\{\Delta \in B(x, y): L(E(\Delta)) \prec L(\bar{M}), D(\Delta, L, \prec)=\emptyset\}} q^{l(\Delta)} .
$$

Then it is clear that

$$
\begin{equation*}
R_{\prec}=h_{\prec} f_{\prec} \quad \text { and } \quad R_{\prec^{\prime}}=h_{\prec} g_{\prec} \tag{5.28}
\end{equation*}
$$

in the incidence algebra of $[e, y]$. We claim that $f_{\prec}(x, y)=g_{\prec}(x, y)$ for all $x \leq y$. In fact, by (5.28) and our induction hypothesis we have that

$$
\begin{equation*}
f_{\prec}(z, M y)=\left(h^{-1} R_{\prec}\right)(z, M y)=\left(h^{-1} R_{\prec}\right)(z, M y)=g_{\prec}(z, M y) \tag{5.29}
\end{equation*}
$$

for all $z \leq M y$ and the claim follows by (5.24) and (5.25).
Therefore, by (5.28), we have

$$
R_{\prec}(x, y)=R_{\prec^{\prime}}(x, y)
$$

and, since $\left(\prec_{i}\right)^{i}=\prec^{i}$

$$
R_{\prec_{i}}(x, y)=R_{\prec^{\prime}}(x, y) .
$$

Now notice that $f_{\prec_{i}}(x, z)=R_{\prec_{i}}(x, z)$ for all $x, z \leq y$ (since $L\left(\bar{M}_{i}\right) \prec_{i} L(E(\Delta))$ is an empty condition) and hence, by (5.24) and (5.29)

$$
R_{\prec_{i}}(x, y)= \begin{cases}R_{\prec_{i}}(M x, M y), & \text { if } M x \triangleleft x \\ R_{\prec_{i}}(M x, M y)+q R_{\prec_{i}}(x, M y), & \text { otherwise }\end{cases}
$$

and the thesis follows by Corollary 0.5.3 and our induction hypothesis.

Now fix $v \in W$, an $R$-regular sequence $\mathcal{M}$ for $v$, a reflection labeling $L$ : $R_{M} \rightarrow T^{\prime}$ and a reflection ordering $\prec$ on $T^{\prime}$. Let $\Delta \in B(x, y)$, where $x \leq$ $y \leq v$. We define the descent composition of $\Delta$ with respect to $\prec$ to be the unique composition $\mathcal{C}(\Delta, L, \prec):=\left(b_{1}, \ldots, b_{j}\right)$ such that $b_{1}+\ldots+b_{j}=l(\Delta)$ and
$D(\Delta, L, \prec)=\left\{b_{1}, b_{1}+b_{2}, \ldots, b_{1}+\ldots+b_{j-1}\right\}$.
For $x, y \leq v$, and $\alpha \in C$, we let

$$
\begin{equation*}
c_{\alpha}(x, y):=\mid\left\{\Delta \in B(x, y): l(\Delta)=|\alpha| \text { and } \mathcal{C}(\Delta, L, \prec) \geq_{c} \alpha\right\} \mid, \tag{5.30}
\end{equation*}
$$

and

$$
\begin{equation*}
b_{\alpha}(x, y):=\mid\{\Delta \in B(x, y): l(\Delta)=|\alpha| \text { and } \mathcal{C}(\Delta, L, \prec)=\alpha\} \mid \tag{5.31}
\end{equation*}
$$

Note that these definitions imply that

$$
\begin{equation*}
c_{\alpha}(x, y)=\sum_{\beta \geq_{c} \alpha} b_{\beta}(x, y) \tag{5.32}
\end{equation*}
$$

for all $x, y \leq v$ and $\alpha \in C$, and that

$$
\begin{equation*}
c_{\alpha}(x, y)=b_{\alpha}(x, y)=\mid\{\Delta \in B(x, y): l(\Delta)=|\alpha| \text { and } D(\Delta, L, \prec)=\emptyset\} \mid \tag{5.33}
\end{equation*}
$$

if $l(\alpha)=1$.
The proof of the following result is analogous to that of Proposition 4.4 of [13] and is therefore omitted.

Proposition 5.3.5 Let $x \leq y \leq v$, and $\alpha \in C$. Then

$$
c_{\alpha}(x, y)=\sum_{\left(x_{0}, \ldots, x_{r}\right) \in C_{r}(x, y)} \prod_{j=1}^{r}\left[q^{\alpha_{j}}\right]\left(\widetilde{R}_{x_{j-1}, x_{j}}\right)
$$

where $C_{r}(x, y)$ denotes the set of all chains of length $r$ from $x$ to $y$, and $r:=$ $l(\alpha) \square$

We can now state and prove the second main result of this section, which generalizes the main result of [14] (Theorem 7.2). Recall the definition of the polynomials $\Psi_{\alpha}(q)$ and $\Upsilon_{\beta}(q)$ from Subsection 5.3.2.

Theorem 5.3.6 Let $v \in W, \mathcal{M}$ be an $R$-regular sequence for $v, L: R_{\mathcal{M}} \rightarrow T^{\prime}$ be a reflection labeling and $\prec$ be a reflection ordering on $T^{\prime}$. Then, for all $x \leq y \leq v$

$$
\begin{equation*}
P_{x, y}(q)-q^{l(x, y)} P_{x, y}\left(\frac{1}{q}\right)=\sum_{\Delta \in B(x, y)} q^{\frac{l(x, y)-l(\Delta)}{2}} \Upsilon_{\mathcal{C}(\Delta, L, \prec)}(q) \tag{5.34}
\end{equation*}
$$

Proof. From Theorem 5.3.1 and Propositions 5.3.2 and 5.3.5 we have that

$$
\begin{aligned}
P_{x, y}(q)-q^{l(x, y)} P_{x, y}\left(\frac{1}{q}\right) & =\sum_{\mathcal{C} \in C(x, y)}(-1)^{l(\mathcal{C})} R_{\mathcal{C}}(q) \\
& =\sum_{\alpha \in C}(-1)^{l(\alpha)} q^{\frac{l(x, y)-|\alpha|}{2}} \Psi_{\alpha}(q) c_{\alpha}(x, y) .
\end{aligned}
$$

On the other hand, from (5.32) and (5.21) we obtain

$$
\begin{aligned}
\sum_{\alpha \in C_{n}}(-1)^{l(\alpha)} \Psi_{\alpha}(q) c_{\alpha}(x, y) & =\sum_{\alpha \in C_{n}}(-1)^{l(\alpha)} \Psi_{\alpha}(q) \sum_{\beta \succeq_{c} \alpha} b_{\beta}(x, y) \\
& =\sum_{\beta \in C_{n}} b_{\beta}(x, y) \sum_{\alpha \preceq_{c} \beta}(-1)^{l(\alpha)} \Psi_{\alpha}(q) \\
& =\sum_{\beta \in C_{n}} b_{\beta}(x, y) \Upsilon_{\beta}(q)
\end{aligned}
$$

for all $n \in \mathbb{P}$. Therefore we conclude that

$$
P_{x, y}(q)-q^{l(x, y)} P_{x, y}\left(\frac{1}{q}\right)=\sum_{\beta \in C} q^{\frac{l(x, y)-|\beta|}{2}} \Upsilon_{\beta}(q) b_{\beta}(x, y),
$$

which, by (5.31), is equivalent to (5.34).

In the same way as Theorem 7.3 is deduced from Theorem 7.2 in [14] one obtains the following result from Theorem 5.3.6. Given $n \in \mathbb{Z}$ and $A \subseteq \mathbb{Z}$ we let $n-A:=\{n-a: \quad a \in A\}$. Recall our notations concerning lattice paths from Subsection 5.3.2..

Corollary 5.3.7 Let $v \in W, \mathcal{M}$ be an $R$-regular sequence for $v, L: R_{\mathcal{M}} \rightarrow T^{\prime}$ be a reflection labeling and $\prec$ be a reflection ordering on $T^{\prime}$. Then, for all $x \leq y \leq v$,

$$
P_{x, y}(q)=\sum_{(\Gamma, \Delta)}(-1)^{\Gamma \geq 0+d_{+}(\Gamma)} q^{\frac{l(x, y)+\Gamma(l(\Gamma))}{2}}
$$

where the sum is over all pairs $(\Gamma, \Delta)$ such that $\Gamma$ is a lattice path, $\Delta \in B(x, y)$, $l(\Gamma)=l(\Delta), N(\Gamma)=l(\Delta)-D(\Delta, L, \prec)$, and $\Gamma(l(\Gamma))<0$.

## Chapter 6

## Special matchings of $\mathfrak{S}(n)$ form a Coxeter group

The proof of Lusztig's conjecture for lower Bruhat intervals (Corollary 4.4.8) uses the fundamental concept of special matchings of a partially ordered set, and follows from the study of all possible commutation relations between two such matchings. In this chapter we study with much more detail the relations between special matchings of intervals of the form $[e, v]$, where $v \in \mathfrak{S}(n)$. In fact, the main result of this chapter (Theorem 6.2.1) is that all the possible relations between special matchings are consequences of the commutation relations among them. Or, which is the same, it states that the group $\widehat{W}_{v}$ generated by the set $S_{v}$ of all the special matchings of a permutation $v$ is again a Coxeter group with $S_{v}$ as set of Coxeter generators. Furthermore the Coxeter system ( $\widehat{W}_{v}, S_{v}$ ) is isomorphic to a direct product of symmetric groups.

### 6.1 The commutation graph

We start this section with two technical lemmas that will later be needed. Lemma 6.1.1 is in the spirit of Lemma 1.1.1.

Lemma 6.1.1 Let $(W, S)$ be any Coxeter system and let $J, K \subseteq S$ with $J \cap$ $K=\emptyset$. Suppose that $w=w_{j} w_{k}$, with $w_{j} \in W_{J}$ and $w_{k} \in W_{K}$, and that $s_{j} \in J \cap D_{R}(w)$. Then $s_{j} \in D_{R}\left(w_{j}\right)$ and $s_{j}$ commutes with every letter in $w_{k}$.

Proof. We proceed by induction on $l\left(w_{k}\right)$, the assertion being clear if $l\left(w_{k}\right)=0$. So suppose $l\left(w_{k}\right) \geq 1$ and let $s \in D_{R}\left(w_{k}\right)$. By the Lifting Lemma (Lemma 0.3.4), $s_{j} \in D_{R}(w s)$, and we can consider the factorization $w s=w_{j} w_{k} s$, with $l\left(w_{k} s\right)<$ $l\left(w_{k}\right)$. So, by induction hypothesis, $s_{j}$ commutes with every letter in $w_{k} s$, namely with every letter in $w_{k}$ except at most $s$. Suppose, by contradiction, that $s_{j}$ does not commute with $s$. By Lemma 0.3.5, $w$ admits a reduced expression of the form $w^{\prime} \alpha_{s, s_{j}}$ where $\alpha_{s, s_{j}}$ has more than two letters. Hence $s \leq w s$ and this forces $s \leq w_{k} s$; but this is a contradiction because we have already proved that $s_{j}$ commutes with every letter in $w_{k} s$.
Clearly, a dual version of Lemma 6.1.1 holds.

Lemma 6.1.2 Let $(W, S)$ be any Coxeter system, and let $w \in W$ and $K \subseteq S$. Suppose that $s \in D_{L}(w)$ but $s \notin K$. Then $s \in D_{L}\left({ }^{K} w\right)$.

Proof. Recall the factorization of Proposition 0.3.7: $w=w_{K}{ }^{K} w$. We proceed by induction on $l\left(w_{K}\right)$, the assertion being clear if $l\left(w_{K}\right)=0$.
Suppose $l\left(w_{K}\right) \geq 1$ and let $s^{\prime} \in D_{L}\left(w_{K}\right)$. By the Lifting Lemma (Lemma 0.3.4), $s \in D_{L}\left(s^{\prime} w\right)$, and we can consider the factorization $s^{\prime} w=s^{\prime} w_{K}{ }^{K} w$, with $l\left(s^{\prime} w_{K}\right)<l\left(w_{K}\right)$. So, by induction hypothesis, $s \in D_{L}\left({ }^{K} v\right)$.
Clearly, a dual version of Lemma 6.1.2 holds.

We also need the following result about the length of dihedral intervals in the symmetric group.

Proposition 6.1.3 Let $u, v \in \mathfrak{S}(n), u \leq v$, be such that the interval $[u, v]$ is dihedral. Then $l(u, v) \leq 3$.

Proof. By Lemma 4.1.1 it follows that the group $W^{\prime}$ generated by the set of reflections $\left\{a b^{-1}: u \leq a \triangleleft b \leq v\right\}$ is a dihedral reflection subgroup. By Theorem 1.4 of [32], it follows that the interval $[u, v]$ is isomorphic, as a partially ordered set, to a subset of $W^{\prime}$. The statement follows since dihedral reflections subgroups of the symmetric group are of length at most 3 .

Remark. In general, Proposition 6.1.3 can be false even if all the entries of the Coxeter matrix are $\leq 3$. A counterexample can be found even in $\widetilde{A}_{2}$, the Coxeter group of Coxeter generators $s_{1}, s_{2}, s_{3}$ with $m\left(s_{i}, s_{j}\right)=3$ for all $i \neq j$. In fact, for example, $\left[s_{1} s_{2} s_{3}, s_{2} s_{1} s_{3} s_{2} s_{1} s_{3} s_{2}\right.$ ] is a dihedral interval of length 4.

From now on we call a dihedral interval of length 1, 2 and 3 respectively a segment, a square and a hexagon.

Proposition 6.1.4 Let $v \in \mathfrak{S}(n), M$ and $N$ be two special matchings of $v$, and let $u_{0} \leq v$.

1. If $\langle M, N\rangle\left(u_{0}\right)$ is a hexagon then $\langle M, N\rangle(u)$ is a hexagon for all $u \leq v$.
2. If $\langle M, N\rangle\left(u_{0}\right)$ is either a segment or a square then $\langle M, N\rangle(u)$ is a segment or a square for all $u \leq v$.

Proof. Actually 1. and 2. are equivalent by Lemma 4.2.2 and Proposition 6.1.3. Let us prove 1.
With no lack of generality, we can suppose that $u_{0}$ is the top element of an orbit. We first prove the statement for $u \leq u_{0}$ by induction on $l\left(u_{0}\right)$. Suppose that there exists $u_{1} \triangleleft u_{0}, u_{1} \notin\left\{M\left(u_{0}\right), N\left(u_{0}\right)\right\}$, such that $u \leq u_{1}$. Then, by Proposition 4.2.3, $\langle M, N\rangle\left(u_{1}\right)$ is a hexagon and we can conclude by our induction hypothesis. If such $u_{1}$ does not exist, by Corollary 4.1.3, $[u, v]$ is a dihedral interval containing $M\left(u_{0}\right)$ and $N\left(u_{0}\right)$. Then, by Theorem 4.1.2 and Proposition 6.1.3, $u \in\langle M, N\rangle\left(u_{0}\right)$ and we are done. In particular we have that $\langle M, N\rangle(e)$ is a hexagon. An upside-down argument with $u_{0}=e$ shows that $\langle M, N\rangle(u)$ is a hexagon for all $u \geq e$ and the proof is complete.

Now we can conclude that the commutation rules of special matchings really look like the Coxeter relations for the symmetric group.

Corollary 6.1.5 Let $v \in \mathfrak{S}(n), M$ and $N$ be two special matchings of $v$. Then either $M N=N M$ or $M N M=N M N$.

Proof. It is straightforward by Proposition 6.1.4.

Now we focus our attention to non-commuting pairs of special matchings. To see that two special matchings $M$ and $N$ do not commute it is enough to check that $M N(e) \neq N M(e)$, by Proposition 6.1.4. It follows that any special matching does not commute with at most 4 other special matchings.

| special matchings | does not commute with |
| :---: | :---: |
| $\lambda_{i}$ | $\lambda_{i-1}, \lambda_{i+1}, l_{i-1}, r_{i+1}$ |
| $\rho_{i}$ | $\rho_{i-1}, \rho_{i+1}, r_{i-1}, l_{i+1}$ |
| $l_{i}$ | $r_{i-1}, r_{i+1}, \lambda_{i+1}, \rho_{i-1}$ |
| $r_{i}$ | $l_{i-1}, l_{i+1}, \lambda_{i-1}, \rho_{i+1}$ |

We shall see that the situation is actually much simpler. We define the commutation graph of the special matchings of $v$ to be the graph $G=(V, E)$ where $V$ is the set of special matchings of $v$ and $E$ is the set of non-commuting pairs of special matchings. For what we have proved so far, if $v \in \mathfrak{S}(7)$, its commutation graph can be obtained from the graph in Figure 6.1 by deleting some vertices and the corresponding adjacent edges. Note that the special matchings $l_{1}, l_{6}$, $r_{1}$, and $r_{6}$ do not appear in this graph since they are necessarily also of type $\lambda$ or $\rho$.


Figure 6.1: Special matchings in $\mathfrak{S}(7)$

Lemma 6.1.6 Let $v \in \mathfrak{S}(n)$.

1. If $l_{i}, r_{i+1}$ are both special matchings of $v$, then $v=v^{\prime} s_{i} s_{i+1} s_{i}$, with $v^{\prime} \in$ $\mathfrak{S}(n)_{S \backslash\{i, i+1\}}$.
2. If $r_{i}, l_{i+1}$ are special matchings of $v$, then $v=s_{i} s_{i+1} s_{i} v^{\prime \prime}$, with $v^{\prime \prime} \in$ $\mathfrak{S}(n)_{S \backslash\{i, i+1\}}$.

Proof. We prove only 1. because 2. is its dual statement.
Let $u \in[e, v]$ be such that $l_{i}(u)>u$ and $r_{i+1}(u)>u$. We show that $s_{i+1} \not \leq u$.
Let $J:=[i]$ and $K:=[i, n-1]$, and decompose $u=u_{J}{ }^{J} u$, where, by Corollary 0.7.5, ${ }^{J} u \in \mathfrak{S}(n)_{K}$. We have $s_{i+1} s_{i+2} \not \not \leq{ }^{J} u$, otherwise $s_{i} s_{i+1} s_{i+2} \leq l_{i}(u) \leq$
$v$ which contradicts 2. of Corollary 0.7.4. Hence, if $s_{i+1} \leq{ }^{J} u$, we have that ${ }^{J} u=u_{1} s_{i+1} u_{2}$ with $u_{1} \in \mathfrak{S}(n)_{[i+2, n-1]}$ and $u_{2} \in \mathfrak{S}(n)_{\{i\}}$. So $u=u_{J}{ }^{J} u=$ $u_{J} u_{1} s_{i+1} u_{2}$ and hence, by Corollary 0.7.5, $r_{i+1}(u)=u_{1} s_{i+1} u_{J} s_{i+1} u_{2}$ which implies $s_{i} s_{i-1} \not \leq u_{J}$, by Corollary 0.7.4. Since $l_{i}(u) \triangleright u$ this forces $s_{i} \not \leq u_{J}$ and hence $r_{i+1}(u)<u$.

A symmetric argument shows that $s_{i} \not \leq u$.
Note that, since $\left\langle l_{i}, r_{i+1}\right\rangle(e)$ is a hexagon all the orbits of the group $\left\langle l_{i}, r_{i+1}\right\rangle$ are hexagons by Proposition 6.1.4. Suppose $u$ is the bottom element of the hexagon containing $v$ so that $v=l_{i} r_{i+1} l_{i}(u)$. We know that $u=u_{1} u_{2}$ with $u_{1} \in \mathfrak{S}(n)_{[i-1]}$ and $u_{2} \in \mathfrak{S}(n)_{[i+2, n-1]}$. Then

$$
\begin{aligned}
v & =l_{i} r_{i+1} l_{i}(u)=l_{i} r_{i+1}\left(u_{1} s_{i} u_{2}\right)=l_{i}\left(u_{2} s_{i+1} u_{1} s_{i}\right) \\
& =u_{1} s_{i} u_{2} s_{i+1} s_{i}=u_{1} u_{2} s_{i} s_{i+1} s_{i}
\end{aligned}
$$

and we are done.

Theorem 6.1.7 Let $v \in \mathfrak{S}(n)$.

1. If $l_{i}$ and $r_{i+1}$ are two special matchings of $v$ then

is a connected component of the commutation graph of the special matchings of $v$.
2. If $r_{i}$ and $l_{i+1}$ are two special matchings of $v$ then

is a connected component of the commutation graph of the special matchings of $v$.

Proof. We prove only 1. because 2 . is its dual statement.
Figure 6.2 shows all possible neighbors of the special matchings $l_{i}$ and $r_{i+1}$ in the commutation graph (see Figure 6.1).

Recall that, by Lemma 6.1.6, $v=v^{\prime} s_{i} s_{i+1} s_{i}$, with $v^{\prime} \in \mathfrak{S}(n)_{S \backslash\{i, i+1\}}$.
If $\lambda_{i+1}$ is a special matching of $v$ then, by Lemma 6.1.1, $s_{i+2} \not \leq v$ which forces


Figure 6.2: Neighbors of $l_{i}$ and $r_{i+1}$
$\lambda_{i+1}=r_{i+1}$. If $\lambda_{i}$ is a special matching of $v$ then, by Lemma 6.1.1, $s_{i-1} \not \leq v$ which implies $\lambda_{i}=l_{i}$.
Again by Lemma 6.1.1, we have that $\rho_{i-1}$ and $\rho_{i+2}$ are not special matchings of $v$.

Let us check that $l_{i+2}$ and $r_{i-1}$ are not special matchings of $v$. We show it for $l_{i+2}$, the same argument being valid also for $r_{i-1}$. Suppose, by contradiction, that $l_{i+2}$ is a special matching. By 2. of Lemma 6.1.6 we have that $v=$ $v^{\prime} s_{i+1} s_{i+2} s_{i+1}$ and $v=s_{i+1} s_{i+2} s_{i+1} v^{\prime \prime}$, with $v^{\prime}, v^{\prime \prime} \in \mathfrak{S}(n)_{S \backslash\{i+1, i+2\}}$. But these two decompositions of $v$ are incompatible, since from the second we have $i+1 \in D_{L}(v)$ which forces, from the first decomposition, $s_{i+2} \not \leq v^{\prime}$. But this is a contradiction with the second one.

We go on in our analysis of the commutation graph by showing another forbidden configuration.

Theorem 6.1.8 Let $v \in \mathfrak{S}(n)$.

1. The configuration

is forbidden in the commutation graph of the special matchings of $v$.
2. The configuration

is forbidden in the commutation graph the special matchings of $v$.

Proof. We prove only 1 . because 2. is its dual statement.
By contradiction, suppose that $\lambda_{j-1}, r_{j}$ and $\rho_{j+1}$ are all special matchings of $v$. Let $J=[j]$ and $K=[j, n-1]$, and decompose $v=v_{K}{ }^{K} v$. We claim that $s_{j} \not \mathbf{K}^{K} v$. In fact ${ }^{K} v s_{j+1} \notin{ }^{K} W$ since ${ }^{K} W \cap[e, v] \subseteq W_{J} \cap[e, v]$ by 2. of Corollary 0.7.5. Then, by the definition of ${ }^{K} W$, there exists $k \in K$ such that $k \in D_{L}\left({ }^{K} v s_{j+1}\right)$. Necessarily $k=j$ or $k=j+1$ as ${ }^{K} v \in W_{J}$.

If $k=j$ we have


By the lifting lemma (Lemma 0.3.4) we should have ${ }^{K} v s_{j+1}=s_{j}{ }^{K} v$. But this is not possible since $s_{j+1} \not \leq s_{j}{ }^{K} v$.

If $k=j+1$ we have that

and hence, by the lifting lemma, ${ }^{K} v s_{j+1}=s_{j+1}{ }^{K} v$ which implies that $s_{j} \not \leq{ }^{K_{v}}$ by Lemma 6.1.1. So the claim is proved.

A similar argument applied to $v^{-1}$ and to $J$, together with Corollary 0.7.5, provides that either $s_{j} \not \leq v_{K}$ or $v_{K}=v^{\prime} s_{j}$ with $s_{j} \not \leq v^{\prime}$. But since $r_{j}$ is a special matching of $v$ we must be in the last situation. So we have

$$
v=v^{\prime} s_{j}{ }^{K} v
$$

with $v^{\prime} \in W_{[j+1, n-1]}$ and ${ }^{K} v \in W_{[j-1]}$. All these conditions are in contradiction with Lemma 6.1.1 (e.g. apply Lemma 6.1 .1 to $j+1 \in D_{R}(v)$ ), and the proof is complete.

The next result shows us how a connected component of the commutation graph looks like.

Theorem 6.1.9 Let $v \in \mathfrak{S}(n)$. Let $\lambda_{i}, \lambda_{i+1}, \ldots, \lambda_{j}$ with $j \geq i$ be special matchings of $v$ and suppose that $\lambda_{i-1}$ and $\lambda_{j+1}$ are not special matchings of $v$. Then their connected component in the commutation graph is a subgraph of


Dually, if $\rho_{i}, \ldots, \rho_{j}$ are special matchings of $v$ and $\rho_{i-1}$ and $\rho_{j+1}$ are not, then their connected component in the commutation graph is a subgraph of


Proof. We only prove the first statement.
Observe that if $\lambda_{k}, \lambda_{k+1}$ and $r_{k+1}$ are special matchings of $v$ for some $k$ then $r_{k+1}=\lambda_{k+1}$. In fact, if $k, k+1 \in D_{L}(v)$ then, by Lemma 0.3.5, $v=s_{k} s_{k+1} s_{k} v^{\prime}$ with $l(v)=l\left(v^{\prime}\right)+3$. But then, if $r_{k+1}$ is a special matching of $v$, we have $s_{k+2} \not \leq v^{\prime}$, otherwise $s_{k} s_{k+1} s_{k+2} \leq v$, which is not possible by Corollary 0.7.4. Hence $r_{k+1}=\lambda_{k+1}$. Similarly, if $\lambda_{k}, \lambda_{k+1}$ and $l_{k}$ are special matchings of $v$ for some $k$ then $l_{k}=\lambda_{k}$.

Now we look at the possible neighbors of $\lambda_{i}$ in the commutation graph. If $l_{i-1}$ is not a special matching there is nothing to prove, because $\lambda_{i}$ is adjacent only to $\lambda_{i+1}$. If $l_{i-1}$ is a special matching, $r_{i-2}$ and $r_{i}$ cannot be special matchings of $v$ by Theorem 6.1.7, and $\rho_{i-2}$ cannot be a special matching of $v$ by 2. of Theorem 6.1.8. The analysis of the commutation graph around $\lambda_{j}$ is similar and it is left to the reader.

Example. Let, for $3 \leq i<j \leq n-3, v=s_{i-2} s_{j+2} w$ where $w$ is the longest element in the parabolic subgroup $W_{[i, j+1]}$. Then


is actually a connected component in the commutation graph of the special matchings of $v$.

Our next goal is to understand if there is any further relation among the special matchings of a permutation $v$, other than the commutation relations. In other words we want to understand if the group generated by the special matchings is the Coxeter group whose Coxeter diagram is our commutation graph or a proper quotient of it. We will see that there are no further relations.

Lemma 6.1.10 Let $i<j$. Suppose that

is a subgraph of a connected component of the commutation graph. Then the other connected components having at least a special matching indexed in $[i-1, j]$ do not have vertices with index smaller than $i-1$.
Clearly, dual statements hold if we change the subgraph with one of the following

$\lambda_{j} \xrightarrow{r_{j+1}}$
$\rho_{i+1} \rho_{i} \xlongequal{\square}$


Proof. Let us check all other possible special matchings indexed by $i-1$. By the proof of Theorem 6.1.9, if $\lambda_{i-1}$ is a special matching, then $\lambda_{i-1}=l_{i-1}$. Let $J=[i-1]$ and decompose $v=v_{J}{ }^{J} v$. Since $l_{i-1}$ is a special matching, we have that $s_{i-1} \in D_{R}\left(v_{J}\right)$. We can assume that $s_{i-2} \leq v_{J}$, otherwise we cannot have special matchigs indexed by $i-2$ and the result would be trivial. If $r_{i-1}$ is a special matching then $s_{i-2} s_{i-1} s_{i} \not \leq v$ by Corollary 0.7.4. Hence necessarily $s_{i} \not \leq{ }^{J} v$. But this is in contradiction with the fact that $\lambda_{i}$ is a special matching and hence $r_{i-1}$ is not a special matching. So the only other possible special matching indexed by $i-1$ is $\rho_{i-1}$.
Note that the possible neighbours of $\rho_{i-1}$ indexed by $i-2$ are $\rho_{i-2}$ and $r_{i-2}$. But $\rho_{i-2}$ and $r_{i-2}$ are not special matchings of $v$ because, otherwise, they would be in the same connected component of $l_{i-1}$ and this is in contradiction with Theorem 6.1.9.

Lemma 6.1.11 Let $i<j$. If

is a connected component of the commutation graph and $C$ is another component with a special matching indexed by $i$, then this special matching is of type $\rho$ or $r$ and if it is of type $r, C$ does not contain any special matching with index smaller than $i$.
Clearly, a dual statement holds.
Proof. We already know that if $l_{i}$ is a special matching, then it is necessarily equal to $\lambda_{i}$. If $\rho_{i}$ is a special matching there is nothing to prove. So suppose
that $r_{i}$ is a special matching. If $l_{i-1}$ or $\lambda_{i-1}$ are special matchings, then they would be in the same connected component of $\lambda_{i}$, which is a contradiction; so in the connected component of $r_{i}$ there are no special matchings with index smaller than $i$.

We introduce an equivalence relation on the set of connected components of the commutation graph. We say that two connected components $C$ and $C^{\prime}$ are in the same isotypical component if there exists a sequence $C=C_{0}, C_{1}, \ldots, C_{t}=$ $C^{\prime}$ of connected components such that, for all $i \in[t], C_{i-1}$ and $C_{i}$ contain at least one special matching with the same index. Then Lemmas 6.1.10 and 6.1.11 tells us that special matchings of type $l$ and $r$ have external indices in isotypical components.

Corollary 6.1.12 Let I be an isotypical component of the commutation graph and suppose that all the special matchings in $I$ are indexed in $[i, j]$. Then all the special matchings of $I$ indexed in $[i+1, j-1]$ are of type $\lambda$ or $\rho$.

Proof. It follows directly from Lemma 6.1.10 and 6.1.11.
This is an example of how an isotypical component looks like.


### 6.2 The Coxeter group $\widehat{W}_{v}$

Given $v \in \mathfrak{S}(n)$, we define $\widehat{W}_{v}$ to be the group generated by the set $S_{v}$ of all special matchings of $v$. In this section, we analyze the structure of the group $\widehat{W}_{v}$. Our goal is to show that the pair $\left(\widehat{W}_{v}, S_{v}\right)$ is again a Coxeter system.

Theorem 6.2.1 Let $v \in \mathfrak{S}(n), S_{v}$ be the set of all special matchings of $v$ and $\widehat{W}_{v}$ be the group generated by $S_{v}$. Then

$$
\left(\widehat{W}_{v}, S_{v}\right)
$$

is a Coxeter system isomorphic to a direct product of symmetric groups.
Proof. Let $p$ be a word in the alphabet of the special matchings of $v$ such that $p(u)=u$ for all $u \leq v$ (in other words, $p$ is the identity in $\widehat{W}_{v}$ ). The result will follow if we show that we can obtain the empty word from $p$ using only braid moves either of the form $M N M \leftrightarrow N M N$ (if $M$ and $N$ do not commute), or of the form $M N \leftrightarrow N M$ (if $M$ and $N$ do commute), and nil moves of the form $M M=\emptyset$. Suppose that $I$ is an isotypical component whose set of indices is $[i, j]$. Then, by Corollary 6.1.12, after commutation of some letters, we may suppose that $p=p_{1} p_{2} p_{3}$, where:

- $p_{1}$ is a word in $h_{i}, \lambda_{i+1} \ldots, \lambda_{j-1}, h_{j}$ with $h_{i}$ equal to $l_{i}$ or $\lambda_{i}$ and $h_{j}$ equal to $r_{j}$ or $\lambda_{j}$;
- $p_{2}$ is a word in $k_{i}, \rho_{i+1}, \ldots, \rho_{j-1}, k_{j}$, with $k_{i}$ equal to $r_{i}$ or $\rho_{i}$ and $k_{j}$ is equal to $l_{j}$ or $\rho_{j}$;
- $p_{3}$ is a word involving special matchings which are not indexed in $[i, j]$.

It is clear that it is enough to prove our claim for $p_{1}$ and $p_{2}$, the general result following by induction on the number of isotypical components. These conditions imply that $p_{1} p_{2}=p_{3}^{-1}$. In particular we have $p_{1} p_{2}(e)=p_{3}^{-1}(e)$. But $p_{1} p_{2}(e) \in \mathfrak{S}(n)_{[i, j]}$ and $p_{3}^{-1}(e) \in \mathfrak{S}(n)_{[1, i-1] \cup[j+1, n-1]}$ and hence $p_{1} p_{2}(e)=$ $p_{3}^{-1}(e)=e$. Moreover, for all $s_{h} \leq v$ we have $p_{1} p_{2}\left(s_{h}\right) \in \mathscr{S}(n)_{[i, j] \cup\{h\}}$ and $p_{3}^{-1}\left(s_{h}\right) \in \mathfrak{S}(n)_{[i, i-1] \cup[j+1, n-1] \cup\{h\}}$ and hence $p_{1} p_{2}\left(s_{h}\right) \in\left\{e, s_{h}\right\}$. Since $p_{1} p_{2}$ is a bijection we have $p_{1} p_{2}\left(s_{h}\right)=s_{h}$.
We firstly deal with the case $i=j$. Let $\pi:=p_{1} p_{2}$. We can clearly assume that $\pi$ is a subword of $\lambda_{i} \rho_{i} l_{i} r_{i}$ of even length, since $\pi(e)=e$. If $s_{i-1} \leq v$ and $s_{i+1} \not \leq v$, then $l_{i}=\rho_{i}$ and $r_{i}=\lambda_{i}$, so $\pi$ is a subword of $\lambda_{i} \rho_{i}$. But $\lambda_{i} \rho_{i}\left(s_{i-1}\right) \neq s_{i-1}$ and
hence $\pi$ is the empty word. If $s_{i-1} \not \leq v$ and $s_{i+1} \leq v$, the proof is very similar. If $s_{i-1} \not \leq v$ and $s_{i+1} \not \leq v$ there is at most one special matching indexed by $i$ and the result follows. So we can assume that $s_{i-1} \leq v$ and $s_{i+1} \leq v$. If $l_{i}$ is a special matching, then $l_{i}\left(s_{i-1} s_{i+1}\right)=s_{i-1} s_{i} s_{i+1}$ which implies, by Corollary 0.7.4, that $r_{i}$ is not a special matching. But the only subword of $\lambda_{i} \rho_{i} l_{i}$ that act as the identity on both $s_{i-1}$ and $s_{i+1}$ is the empty word and we are done. The other possible cases are similar and hence are left to the reader.
So we can assume $i<j$, and we restrict our attention on $P:=\mathfrak{S}(n)_{[i, j]} \cap[e, v]$. Note that for all $u \in P$ we have $h_{i}(u)=s_{i} u, h_{j}(u)=s_{j} u, k_{i}(u)=u s_{i}$, and $k_{j}(u)=u s_{j}$ so that we can "think" of the $h$ and the $k$ as, respectively, $\lambda$ and $\rho$ (and the commutation relations do not change!). Thus, if $s_{i_{1}} \cdots s_{i_{k}}$ is a reduced expression of $p_{2}(e)$, we may obtain, using only the commutation relations, $p_{1}^{-1}=\lambda_{i_{1}} \cdots \lambda_{i_{k}}$ and $p_{2}=\rho_{i_{k}} \cdots \rho_{i_{1}}$, so that $p_{1}^{-1}$ acts on $P$ by multiplying on the left by $u:=s_{i_{1}} \cdots s_{i_{k}}$ (this being a reduced expression) and $p_{2}$ acts on $P$ by multiplying on the right by $u$. Since $p_{2}\left(s_{h}\right)=p_{1}^{-1}\left(s_{h}\right)$ for all $h \in[i, j]$ this implies that $u$ belongs to the center of $\mathfrak{S}(n)_{[i, j]}$. The result follows since the center of $\mathfrak{S}(n)_{[i, j]}$ is trivial.

Example. Let $v=(316425) \in \mathfrak{S}(6)$. Then $v$ admits the following reduced expression $v=s_{2} s_{3} s_{1} s_{5} s_{4} s_{3}$. One may check that the interval $[e, v]$ has exactly 6 distinct special matchings and these are $\lambda_{2}, \lambda_{5}, \rho_{1}, \rho_{3}, \rho_{4}, l_{3}$ and $r_{4}$. Then the commutation graph of $v$ is

and the group $\widehat{W}_{v}$ generated by these special matchings is isomorphic to $\mathfrak{S}(3)^{2} \times$ $\mathfrak{S}(2)^{3}$.
Remark Theorem 6.2.1 cannot be generalized to arbitrary Coxeter groups. Let $W$ be the dihedral group generated by $a$ and $b$ with $m(a, b)>4$ and consider $w=a b a b$. Then $[e, w]$ is a dihedral interval of length 4 . Consider the special matchings in Figure 6.3.


| $\ldots-\ldots-$ | $M_{1}$ |
| :--- | :--- |
| $\ldots \ldots \ldots$ | $M_{2}$ |
| $\ldots-\cdots-\ldots-$ | $M_{3}$ |
| $\ldots \ldots-\ldots-$ | $M_{4}$ |

Figure 6.3: dihedral of length 4

Call $M_{1}$ the dashed matching, $M_{2}$ the dotted matching, $M_{3}$ the dash-dotted matching and $M_{4}$ the dash-dot-dotted matching. Then $M_{4} M_{3} M_{2} M_{1}$ is the identity as application from $[e, w]$ to itself but it is clearly not a Coxeter relation.

## Chapter 7

## Kazhdan-Lusztig polynomials for arbitrary posets

This chapter is organized around the problem of generalizing the definition of $R$-polynomials (and hence $\widetilde{R}$-polynomials) and Kazhdan-Lusztig polynomials to arbitrary posets. We find that, in a certain class of posets, the concept of special matching leads to an entirely poset theoretic definition of Kazhdan-Lusztig and $R$-polynomials. This class of posets, which we call diamonds, includes the lower Bruhat intervals and the new definitions are obviously consistent with the classical definitions.

### 7.1 Zircons

Before introducing the class of diamonds, we introduce a more general class of partially ordered sets, which we call zircons. Given a poset $P$, we say that $M$ is a special matching of an element $w \in P$ if $M$ is a special matching of the Hasse diagram of $\{x \in P: x \leq w\}$. We denote by $S_{w}$ the set of all special matchings of $w$.

Definition 7.1.1 We say that a locally finite ranked poset $Z$ is a zircon if $S_{w}$ is non-empty for all $w \in Z$, $w$ not minimal.

Note that, given a zircon $Z$ with rank function $\rho$, then $|\{z \in Z: z \leq w\}|<\infty$ and $l(\{z \in Z: z \leq w\}) \leq \rho(w)$ for all $w \in Z$. Figure 7.1 shows an example of a zircon.


Figure 7.1: a zircon

Let us prove some properties of zircons.
Proposition 7.1.2 Let $Z$ be a zircon with rank function $\rho$, and let $z \in Z$. Then $|\{x \in Z: x \triangleleft z\}| \leq \rho(z)$.

Proof. We proceed by induction on $\rho(z)$, the cases $\rho(z)=0,1$ being clear. Suppose $\rho(z) \geq 2$. Let $M$ be a special matching of $z$. By definition of special matchings, $M(x) \leq M(z)$ for all $x$ such that $x \triangleleft z, x \neq M(z)$. Thus $|\{x \in Z: x \triangleleft z\}|-1 \leq|\{x \in Z: x \triangleleft M(z)\}|$. But by induction hypothesis, $|\{x \in Z: x \triangleleft M(z)\}| \leq \rho(M(z))=\rho(z)-1$.

Proposition 7.1.3 Let $Z$ be a zircon, $m_{1}$ and $m_{2}$ be two minimal elements in $Z$. Then there does not exist $z \in Z$ such that $z \geq m_{1}$ and $z \geq m_{2}$.

Proof. By contradiction, choose a minimal element $z$ among those greater than both $m_{1}$ and $m_{2}$. By the definition of zircon, there exists a special matching $M$ of $z$. By the Lifting Lemma for special matchings (Lemma 0.7.1), $M(z) \geq m_{1}, m_{2}$. But $M(z) \leq z$ and this is a contradiction.

Corollary 7.1.4 Any connected zircon $Z$ is a graded poset.
Proof. By Proposition 7.1.3, $Z$ has a $\hat{0}$. It remains to prove that, given any $z \in Z,[\hat{0}, z]$ is pure. But a finite ranked poset with $\hat{0}$ and $\hat{1}$ clearly satisfies the properties of a pure poset.

Note that any Coxeter group partially ordered by Bruhat order is a connected zircon. In fact, any Coxeter group $W$ is ranked by the function length and, for all $w \in W$, any right or left descent of $w$ gives a special matching of $w$.

Let us plung into the study of the local structure of zircons.
Proposition 7.1.5 Any interval of length 2 of a zircon $Z$ is a square.
Proof. By contradiction, let $z \in Z$ be an element of smallest rank such that it is the top of an interval $[x, z]$ which is not a square. Let $M$ be a special matching of $z$.
Case i) $[x, z]=\{x, y, z\}$
Necessarily, $M(x) \triangleleft x$ otherwise $M$ would restrict to $[x, z]$ by Lemma 4.2.1, and this is not possible because $|[x, z]|=3$ is odd. By our induction hypothesis, $[M(x), y]=\{M(x), x, a, y\}$ is a square. By the definition of special matching, $a \triangleright M(x)$ implies $M(a) \triangleright x$. Then $M(a) \in[x, z]$ and necessarily $M(a)=y$. Hence $M(z) \neq y$ and, by the Lifting Lemma (Lemma0.3.4) and by induction hypothesis, $[M(x), M(z)]=\{M(x), a, b, M(z)\}$ is a square. $M(x) \triangleleft b$ implies $x \triangleleft M(b)$, hence $M(b) \in[x . z], M(b) \neq y$, which is a contradiction.
Case ii) $|[x, z]|>4$
Suppose that $a, b, c \in[x, z] \backslash\{x, z\}$, all distinct. If $M(z) \in[x, z]$, say $M(z)=$ $a$, then $M(b), M(c) \notin[x, z]$, otherwise by Lemma 4.2.1 $M$ would restrict to $[x, z]$. Hence by the definition of special matching, $a \triangleright M(b), M(c), x$ and $M(x) \triangleleft M(b), M(c), x$. So $[M(x), a]$ is not a square and this is a contradiction by the minimality of $z$. If $M(z) \notin[x, z]$, then by the definition of special matching $M(z) \triangleright M(a), M(b), M(c)$ and $M(x) \triangleleft M(a), M(b), M(c)$. So $[M(x), M(z)]$ is not a square and this is again a contradiction.

Proposition 7.1.6 Let $Z$ be a connected zircon with rank function $\rho$ and let $z \in Z$. Then

1. if $\rho(z)=3$, the poset $[\hat{0}, z]$ is a 2 or 3-krown;
2. if $\rho(z)=4$, the poset $[\hat{0}, z]$ is isomorphic to one of the following posets in $\mathfrak{S}(5)$ :
(a) $\left[e, s_{1} s_{2} s_{3} s_{4}\right]$,
(b) $\left[e, s_{2} s_{1} s_{3} s_{2}\right]$,
(c) $\left[e, s_{1} s_{2} s_{1} s_{3}\right]$,
or it is isomorphic to one of the two posets in Figure 7.2, or it is a dihedral interval of length 4.


Figure 7.2: zircons of length 4

Proof. Let us prove the first statement. Let $M \in S_{z}$. Propositions 7.1.2 and 7.1.5 give bounds for the cardinality of $\{x \in Z: x \triangleleft z\}$, namely:

$$
2 \leq|\{x \in Z: x \triangleleft z\}| \leq 3
$$

Case $|\{x \in Z: x \triangleleft z\}|=3$.
Let $\{x \in Z: x \triangleleft z\}=\left\{a_{1}, a_{2}, a_{3}\right\}$, and $M(z)=a_{1}$. By the definition of special matchings, $M\left(a_{2}\right) \triangleleft a_{1}, a_{2}, M\left(a_{3}\right) \triangleleft a_{1}, a_{3}$. By Proposition 7.1.5, $\left[\hat{0}, a_{2}\right]$ is a square, and necessarily $\left[\hat{0}, a_{2}\right]=\left\{\hat{0}, M(\hat{0}), M\left(a_{2}\right), a_{2}\right\}$. By the definition of special matching, $M(\hat{0}) \triangleleft a_{3}$. Clearly $\{x \in[\hat{0}, z]: \rho(x)=1\}=$ $\left\{M(\hat{0}), M\left(a_{2}\right), M\left(a_{3}\right)\right\}$ because other elements would not be matchable. So $[\hat{0}, z]$ is a 3 -krown.
Case $|\{x \in Z: x \triangleleft z\}|=2$. Let $\{x \in Z: x \triangleleft z\}=\left\{a_{1}, a_{2}\right\}$, and $M(z)=a_{1}$. By the definition of special matchings, $M\left(a_{2}\right) \triangleleft a_{1}, a_{2}$. By Proposition 7.1.5, [0̂, $a_{2}$ ] is a square, and necessarily $\left[\hat{0}, a_{2}\right]=\left\{\hat{0}, M(\hat{0}), M\left(a_{2}\right), a_{2}\right\}$. Clearly $\{x \in[\hat{0}, z]$ : $\rho(x)=1\}=\left\{M(\hat{0}), M\left(a_{2}\right)\right\}$ because other elements would not be matchable. It remains to prove that $M(\hat{0}) \triangleleft a_{1}$ and this follows from the fact that [ $\hat{0}, a_{1}$ ] is a square.
Note that $[0, z]$ is a 2 -krown if $M(0) \leq M(z)$; it is a 3-krown otherwise.

The proof of the second statement is similar to that of the first one. Let $M \in S_{z}$. Again Propositions 7.1.2 and 7.1.5 give bounds for the cardinality of
$\{x \in Z: x \triangleleft z\}$. Now we have

$$
2 \leq|\{x \in Z: x \triangleleft z\}| \leq 4
$$

Case $|\{x \in Z: x \triangleleft z\}|=4$.
Let $\{x \in Z: x \triangleleft z\}=\left\{a_{1}, a_{2}, a_{3}, a_{4}\right\}$, and $M(z)=a_{1}$. By the definition of special matchings, $a_{1} \triangleright M\left(a_{2}\right), M\left(a_{3}\right), M\left(a_{4}\right)$, so $\left[\hat{0}, a_{1}\right]$ is a 3 -krown by the first statement. Suppose that there is another 3 -krown starting from $\hat{0}$, say $\left[\hat{0}, a_{2}\right]$. By the definition of special matchings, we have that $\left[\hat{0}, a_{1}\right] \cap\left[\hat{0}, a_{2}\right]$ is either $\left[\hat{0}, M\left(a_{2}\right)\right]$ or $\left[\hat{0}, M\left(a_{2}\right)\right] \cup\{y\}$, where $\rho(y)=1$. In the first case, by Lemma 7.1.5, $M(\hat{0}) \not \leq M\left(a_{3}\right), M\left(a_{4}\right)$, and so $\left[\hat{0}, a_{3}\right]$ and $\left[\hat{0}, a_{4}\right]$ are also 3 -krowns. Hence (a) holds. In the second case, $M(\hat{0}) \leq M\left(a_{3}\right), M\left(a_{4}\right)$, and [ $\hat{0}, a_{3}$ ] and [ $\hat{0}, a_{4}$ ] are 2 -krowns. Hence (b) holds. It remains to prove that is not possible that [ $\hat{0}, a_{2}$ ], [ $\hat{0}, a_{3}$ ] and $\left[\hat{0}, a_{4}\right]$ are all 2-krowns. This follows by noting that in this case $\left[\hat{0}, a_{2}\right] \cap\left[\hat{0}, a_{3}\right] \cap\left[\hat{0}, a_{4}\right]=\{\hat{0}\}$, and hence there are no possibilities for $M(\hat{0})$.
Case $|\{x \in Z: x \triangleleft z\}|=3$.
Let $\{x \in Z: x \triangleleft z\}=\left\{a_{1}, a_{2}, a_{3}\right\}$, and $M(z)=a_{1}$. Using the same technics, one can see that (c) holds either if $[\hat{0}, M(z)]$ is a 3 -krown or if $[\hat{0}, M(z)]$ is a 2 -krown and $\left[\hat{0}, a_{2}\right]$ is a 3 -krown. Let us analyze the case both $[\hat{0}, M(z)]$ and $\left[\hat{0}, a_{2}\right]$ are 2-krowns. By the definition of special matchings, $M(z) \triangleright M\left(a_{2}\right), M\left(a_{3}\right)$. Set $y \triangleleft a_{2}, y \neq M\left(a_{2}\right)$. Since $[y, z]$ is a square, $y \triangleleft a_{3}$. Now $\{x \in[\hat{0}, z]: \rho(x)=$ $2\}=\left\{M\left(a_{2}\right), M\left(a_{3}\right), y\right\}$ because, for another element $y^{\prime},\left[y^{\prime}, z\right]$ would not be a square. All this leads to the poset to the left in Figure 7.2.
Case $|\{x \in Z: x \triangleleft z\}|=2$.
Let $\{x \in Z: x \triangleleft z\}=\left\{a_{1}, a_{2}\right\}$, and $M(z)=a_{1}$. By the definition of special matchings, $M\left(a_{2}\right) \triangleleft a_{1}, a_{2}$. Choose $c_{1} \triangleleft M\left(a_{2}\right)$ such that $c_{1} \triangleleft M\left(c_{1}\right)$. It exists because $\left[\hat{0}, M\left(a_{2}\right)\right]$ is a square by Proposition 7.1.5. Also $\left[c_{1}, a_{2}\right]$ is a square, and necessarily $\left[c_{1}, a_{2}\right]=\left\{c_{1}, M\left(c_{1}\right), M\left(a_{2}\right), a_{2}\right\}$. By the fact that $\left[M\left(c_{1}\right), z\right]$ is a square, we have that $M\left(c_{1}\right) \triangleleft a_{1}$. Suppose that $\left\{M\left(a_{2}\right), M\left(c_{1}\right)\right\}=\{x \in[\hat{0}, z]$ : $\rho(x)=2\}$ and set $c_{2} \in\{x \in[\hat{0}, z]: \rho(x)=1\}$. Then $M\left(c_{2}\right)=\hat{0}$ and $\{x \in[\hat{0}, z]:$ $\rho(x)=1\}=\left\{c_{1}, c_{2}\right\}$ because other elements would not be matchable. Hence $[\hat{0}, z]$ is dihedral. On the contrary, if there exists $y \in\{x \in[\hat{0}, z]: \rho(x)=2\}$, $y \neq M\left(a_{1}\right), M\left(c_{1}\right)$, we have that $y \triangleleft a_{1}, a_{2}$ because $[y, z]$ is a square. Then $\left[\hat{0}, a_{2}\right]$ is isomorphic to the poset to the right in Figure 7.2 because $\left[\hat{0}, a_{1}\right]$ and $\left[\hat{0}, a_{2}\right]$ must be 3 -krowns by the first statement.

### 7.2 Diamonds

In this section we prove the main result of this chapter. We show that the concept of special matching leads to an entirely poset theoretic definition of $R$-polynomials, $\widetilde{R}$-polynomials and Kazhdan-Lusztig polynomials for a certain class of posets, which we call diamonds.

Definition 7.2.1 We say that a connected zircon $D$ is a diamond if, for all $w \in D$ and for all $(M, N) \in S_{w} \times S_{w}$, there exists a sequence $\left(M_{0}, M_{1}, \ldots, M_{k}\right)$ of special matchings in $S_{w}$ such that:

- $M_{0}=M$
- $M_{k}=N$
- for all $i=0,1, \ldots, k-1$,

$$
\begin{equation*}
\left|\left\langle M_{i}, M_{i+1}\right\rangle(x)\right| \text { divides }\left|\left\langle M_{i}, M_{i+1}\right\rangle(w)\right| \tag{7.1}
\end{equation*}
$$

for all $x \in D, x \leq w$.
Let us do a few simple considerations on diamonds.

1. A diamond $D$ does not necessarily admit special matchings of all the poset. Not only, there exist finite diamonds $D$ of odd cardinality, such as the following trivial one.

2. A diamond does not necessarely avoid $K_{3,2}$, as the following.

3. The hipothesis "connected" in Definition 7.2 .1 is not essential but clearly does not affect the problem of defining Kazhdan-Lusztig polynomials.

We now define the $\widetilde{R}$-polynomials of an arbitrary diamond (throught Definition 7.2.2), and then we prove that they do not depend on the choosen special matching. Maybe this is not the most elegant way, but it is certainly the easiest, and mimics what we did for the Coxeter groups.

Definition 7.2.2 For all $w \in D$, choose a special matching of $[\hat{0}, w]$ and denote it by $N_{w}$. Then, for all $u, w \in D$, we inductively define the $\widetilde{R}$-polynomial $\widetilde{R}_{u, w}(q)$ by the following recursive property:

$$
\widetilde{R}_{u, w}(q)= \begin{cases}\widetilde{R}_{N_{w}(u), N_{w}(w)}(q)+\chi\left(N_{w}(u) \triangleright u\right) q \widetilde{R}_{u, N_{w}(w)}(q), & \text { if } u \leq w, \\ 0 & \text { if } u \not \leq w .\end{cases}
$$

The point is to prove that Definition 7.2.2 is well defined, namely that it does not depend on the family $\left\{N_{w}\right\}_{w \in D}$ of special matchings.

Theorem 7.2.3 Let $D$ be a diamond, $w \in D$, and $M$ be a special matching of w. Then,

$$
\begin{equation*}
\widetilde{R}_{u, w}(q)=\widetilde{R}_{M(u), M(w)}(q)+\chi(M(u) \triangleright u) q \widetilde{R}_{u, M(w)}(q), \tag{7.2}
\end{equation*}
$$

for all $u \leq w$.

Proof. We proceed by induction on $\rho(w)$ the statement being trivial if $\rho(w)=1$. So assume $\rho(w) \geq 2$ and fix $u \leq w$. Let $\left\{N_{w}\right\}_{w \in D}$ be as in definition 7.2.2 and, for brevity, set $N:=N_{w}$. We may clearly assume that $M$ and $N$ satisfy (7.1). Denote by $u_{1}, u_{2}, \ldots, u_{2 m}$ the elements of $\langle M, N\rangle(u)$ indexed so that $u_{i}<u_{j}$ implies $i<j$. Let $F$ be the free $\mathbf{Z}[q]$-module generated by $u_{i}, i \in[2 m]$. We define two module endomorphisms $A, B: F \rightarrow F$ by letting

$$
A\left(u_{i}\right):=M\left(u_{i}\right)+\chi\left(M\left(u_{i}\right) \triangleright u_{i}\right) q u_{i}
$$

and

$$
B\left(u_{i}\right):=N\left(u_{i}\right)+\chi\left(N\left(u_{i}\right) \triangleright u_{i}\right) q u_{i},
$$

for all $i \in[2 m]$. We claim that

$$
\begin{equation*}
\underbrace{\cdots A B A}_{m}=\underbrace{\cdots B A B}_{m} . \tag{7.3}
\end{equation*}
$$

In fact, consider the Coxeter system $(G, S)$, where $S=\{s, t, r\}, m(s, t)=m$ and $m(s, r)=m(t, r)=3$ (where $s:=t$ if $m=1$ ). Let $G^{\prime}$ be the parabolic subgroup $G^{\prime}:=G_{\{s, t\}}, H:=\bigoplus_{x \in G^{\prime}} \mathbf{Z}[q] x$ and $\Phi: F \rightarrow H$ be the unique module isomomorphism such that $\Phi\left(u_{1}\right)=e, \Phi\left(M\left(u_{i}\right)\right)=s \Phi\left(u_{i}\right)$ and $\Phi\left(N\left(u_{i}\right)\right)=$ $t \Phi\left(u_{i}\right)$ for all $i \in[2 m]$. Denote $x_{i}:=\Phi\left(u_{i}\right)$, for all $i \in[2 m]$. Then, by our definitions, the endomorphisms $\alpha, \beta: H \rightarrow H$ defined by

$$
\alpha(x):=s x+\chi(s x \triangleright x) q x
$$

and

$$
\beta(x):=t x+\chi(t x \triangleright x) q x,
$$

for all $x \in G^{\prime}$, satisfy $\Phi \circ A=\alpha \circ \Phi$ and $\Phi \circ B=\beta \circ \Phi$. Hence to prove (7.3) it is enough to show that $\underbrace{\ldots \alpha \beta \alpha}_{m}=\underbrace{\ldots \beta \alpha \beta \beta}_{m}$. For all $g \in G$ and all $h=\sum_{i \in[2 m]} h_{i}(q) x_{i} \in H$ we define $h^{g} \in \mathbb{Z}[q]$ by

$$
h^{g}:=\sum_{i \in[2 m]} h_{i}(q) \widetilde{R}_{x_{i}, g}(q) .
$$

Note that, if $s g \triangleleft g$ then $h^{g}=(\alpha(h))^{s g}$ by Corollary 0.5.3, and similarly if $t g \triangleleft g$ then $h^{g}=(\beta(h))^{t g}$. In particular, if $s g \triangleleft g$ and $t g \triangleleft g$ then

$$
h^{g}=(\underbrace{(\ldots \alpha \beta \alpha}_{k}(h)) \underbrace{\underbrace{}_{k} \text { sts } g}_{k}=(\underbrace{\ldots \beta \alpha \beta}_{k}(h))_{k}^{\ldots \text { tst }} g
$$

for all $k \leq m$. If $k=m$ we deduce that

$$
\begin{equation*}
(\underbrace{\ldots \alpha \beta \alpha}_{m}(h))^{g_{0}}=(\underbrace{\ldots \beta \alpha \beta}_{m}(h))^{g_{0}}, \tag{7.4}
\end{equation*}
$$

for all $h \in H$ and all $g_{0} \in G$ such that $s g_{0} \triangleright g_{0}$ and $t g_{0} \triangleright g_{0}$.

Now fix, for the rest of the proof, $i \in[2 m]$ and let $\underbrace{\ldots \alpha \beta \alpha}_{m}\left(x_{i}\right)=\sum_{j} P_{j}(q) x_{j}$ and $\underbrace{\ldots \beta \alpha \beta}_{m}\left(x_{i}\right)=\sum_{j} Q_{j}(q) x_{j}$. If we let $S_{j}(q):=P_{j}(q)-Q_{j}(q)$ for all $j \in[2 m]$, (7.3) will be proved if we show that $S_{j}(q)=0$ for all $j \in[2 m]$. We prove this
by induction on $j$. Equation (7.4), for $h=x_{i}$, implies that

$$
\begin{equation*}
\sum_{j \in[2 m]} S_{j}(q) \widetilde{R}_{x_{j}, g_{0}}(q)=0 \tag{7.5}
\end{equation*}
$$

for all $g_{0} \in G$ such that $s g_{0} \triangleright g_{0}$ and $t g_{0} \triangleright g_{0}$. If we set $g_{0}=r$ in (7.5) we obtain

$$
S_{1}(q) \widetilde{R}_{e, r}(q)=0
$$

forcing $S_{1}(q)=0$. Now let $j>1$ and suppose that $S_{k}(q)=0$ for $k<j$. If we set $g_{0}=r x_{j}$ (note that $s\left(r x_{j}\right) \triangleright r x_{j}$ and $t\left(r x_{j}\right) \triangleright r x_{j}$ since $r$ does not commute neither with $s$ nor with $t$ ) in (7.5) we have that

$$
S_{j}(q) \widetilde{R}_{x_{j}, r x_{j}}(q)=0
$$

which implies $S_{j}(q)=0$ and the proof of (7.3) is completed.
For $f=\sum_{i} f_{i}(q) u_{i} \in F$ and $w \in W$ we let $f^{w}:=\sum_{i} f_{i}(q) \widetilde{R}_{u_{i}, w}(q)$. Note that in this notation (7.2) can be reformulated as

$$
u^{w}=(A(u))^{M(w)} .
$$

By alternated use of the propety defining $N$ and our induction hypothesis we have

$$
u^{w}=(B(u))^{N(w)}=(A B(u))^{M N(w)}=(\underbrace{\cdots B A B}_{n}(u))_{n}^{\underbrace{\cdots N N(w)}_{n}},
$$

and similarly

$$
(A(u))^{M(w)}=(\underbrace{\cdots A B A}_{n}(u)) \underbrace{\cdots M N M}_{n}(w)
$$

where $2 n=|\langle M, N\rangle(w)|$. The thesis follows from (7.3) since $m$ divides $n$ by the definition of diamonds and $\underbrace{\cdots M N M}_{n}(w)=\underbrace{\cdots N M N}_{n}(w)$.

After Definition 7.2.2, we can clearly define the $R$-polynomials and the Kazhdan-Lusztig polynomials of a diamond by generalizing respectively (5) and Theorem 0.5.8. Hence, given a diamond $D$, for all $u, v \in D$ we let $\widetilde{R}_{u, v}(q)$ be the unique polynomial satisfying

$$
R_{u, v}(q)=q^{\frac{l(u, v)}{2}} \widetilde{R}_{u, v}\left(q^{\frac{1}{2}}-q^{-\frac{1}{2}}\right)
$$

The Kazhdan-Lusztig polynomials of a diamond are defined through the following theorem-definition.

Theorem 7.2.4 Let $D$ be a diamond. Then there is a unique family of polynomials $\left\{P_{u, v}(q)\right\}_{u, v \in D} \subseteq \mathbb{Z}[q]$ satisfying the following conditions:

1. $P_{u, v}(q)=0$ if $u \not \leq v$;
2. $P_{u, u}(q)=1$;
3. $\operatorname{deg}\left(P_{u, v}(q)\right) \leq \frac{1}{2}(\rho(v)-\rho(u)-1)$, if $u<v$;
4. if $u \leq v$, then

$$
q^{\rho(v)-\rho(u)} P_{u, v}\left(\frac{1}{q}\right)=\sum_{u \leq z \leq v} R_{u, z}(q) P_{z, v}(q)
$$

Proof. Straightforward by the restriction on $\operatorname{deg}\left(P_{u, v}(q)\right)$.

The following result proves what one certainly wishes to be true.
Theorem 7.2.5 All Coxeter groups partially ordered by Bruhat order are diamonds.

Proof. Let $(W, S)$ be a Coxeter system and let $M$ and $N$ be two special matching of an element $w \in W$. Suppose first that $[e, w]$ is not dihedral. If $M$ and $N$ are both of type $\lambda$ or $\rho$, then $(M, N)$ satisfies (7.1). Suppose that $M$ is of type $\lambda$, -multiplication for a certain $s \in S$-, $N$ is of type $\rho$, and $s s_{2} s_{3} \cdots s_{r}$ is a reduced expression of $w$. Call $\rho_{r}$ the special matching given by the multiplication to the right for $s_{r}$. Then $\left(M, \rho_{r}, N\right)$ satifies (7.1). If $M$ and $N$ are not both multiplication matchings, then the assertion follows by Theorem 4.4.7.

Now suppose that $[e, w]$ is a dihedral interval of length $n$. The set $S_{w}$ of the special matchings of $w$ is in bijection with the set of all $n$-sequences with entries in $\{l, r\}$, ending with $r$. In fact, for all $i=1, \ldots, n-1$, fix $\{v \in[e, w]: l(v)=i\}=$ $\left\{v_{i, l}, v_{i, r}\right\}$ and send a special matching $M$ to the sequence $\left(x_{n-1}, x_{n-2}, \ldots, x_{1}, r\right)$ where $x_{i}=l$ if $M\left(v_{i, l}\right) \triangleright v_{i, l}, x_{i}=r$ if $M\left(v_{i, r}\right) \triangleright v_{i, r}$. In Figure 7.3, the sequence associated to the dotted special matching is $(l, r, r, l, l, r)$.

Any two such sequences give rise to a composition of $n$, just by looking at the positions where they coincide. For example, the sequences ( $l, l, r, r, r, l, r, l, r)$ and $(r, l, l, l, r, l, l, l, r)$ give rise to the composition $(2,3,1,2,1)$ of 9 since they


Figure 7.3: dihedral of length 6
have same entries in positions $2,5,6,8$, and clearly 9 . Two special matchings satisfy (7.1) if all the terms in the composition associated to them divide the first term. Let us show that there exists a chain of sequences such that

- any two consecutively sequences satisfy this property;
- it starts with $\bar{x}=\left(x_{n-1}, \ldots, x_{1}, r\right)=(r, r, \ldots, r)$;
- it ends with $\bar{y}=\left(y_{n-1}, \ldots, y_{1}, r\right)=(r, \ldots, r, l, r, \ldots, r)$, for all possible positions of the unique $l$.

Then the assertion will follow by transitivity and by the symmetry of the problem. If $l=y_{n-1}$, then the composition associated is $(2,1, \ldots, 1)$ and we can choose the trivial chain of the two sequences. If $l=y_{i} \neq y_{n-1}$, then we can consider the sequence $\bar{z}=(l, r, r, \ldots, r)$ and hence the chain $(\bar{x}, \bar{z}, \bar{y})$, which has the required properties.

The new definitions of $\widetilde{R}$-polinomials, $R$-polynomials and Kazhdan-Lusztig polynomials are obviously consistent. In particular, given $d$ in a diamond $D$,

$$
\begin{equation*}
\widetilde{R}_{\hat{0}, d}(q)=q^{\rho(d)} \tag{7.6}
\end{equation*}
$$

if $[\hat{0}, d]$ is a Boolean algebra. Moreover, given $u, v$ in a diamond $D, u \leq v$, it is straightforward by Theorem 7.2.3 that

$$
\widetilde{R}_{u, v}(q)= \begin{cases}q, & \text { if } \rho(v)-\rho(u)=1 \\ q^{2}, & \text { if } \rho(v)-\rho(u)=2\end{cases}
$$

We say that a poset is n-gon-avoiding if it does not contain a dihedral interval of length $\frac{n}{2}$. We say that a poset is lower n-gon-avoiding if it does not contain a dihedral interval of length $\frac{n}{2}$ containing a minimal element.

Theorem 7.2.6 Let $Z$ be a connected zircon which is both lower 8-gon-avoiding and $K_{3,2}$-avoiding. Suppose that for all $w \in Z, \rho(w) \geq 2$, and for all $M \in S_{w}$ there exists a special matching $M^{\prime} \in S_{w}$ such that $M(w) \neq M^{\prime}(w)$. Then $Z$ is a diamond.

Proof. Note first that Corollary 4.1.3, Proposition 4.2.3 and then Lemma 4.2.5 hold under these hypotheses.
We have to prove that for all $w \in Z$ and for all $(M, N) \in S_{w} \times S_{w}$ there exists a sequence of special matchings in $S_{w}$ satisfying the properties of Definition 7.2.1. We proceed by induction on $\rho:=\rho(w)$, the result being clear if $\rho=1$.

So, assume $\rho \geq 2$. Firstly, we prove that, if $M(w) \neq N(w)$, the sequence $(M, N)$ satisfies (7.1), i.e. $|\langle M, N\rangle(x)|$ divides $|\langle M, N\rangle(w)|$ for all $x \leq w$. So set $2 n:=|\langle M, N\rangle(w)|$, where $n \geq 2$. Let $u \leq w$ and $2 m:=|\langle M, N\rangle(u)|$. We have to prove that $m$ divides $n$ so we may assume $m \geq 2$. By applying Lemma 4.2.5 to $\langle M, N\rangle(w)$ and $\langle M, N\rangle(u)$ we obtain that there exist a lower dihedral interval containing an orbit of cardinality $n$ and a lower dihedral interval containing an orbit of cardinality $m$. Hence $\{m, n\} \subset\{2,3\}$ since $Z$ is lower 8 -gon-avoiding. If $M(\hat{0}) \neq N(\hat{0})$ then, by Lemma 4.2.5, the two dihedral intervals are coincident, which forces $m=n$.
If $M(\hat{0})=N(\hat{0})$ then the two dihedral intervals are not necessarily coincident, but clearly there remains place only for orbits of cardinality 4 . Hence $m=n=2$.

Now suppose that $M(w)=N(w)$. By our hypotheses, there exists a special matching $M^{\prime} \in S_{w}$ such that $M(w) \neq M^{\prime}(w)$. Then by what we have already proved, $\left(M, M^{\prime}, N\right)$ satisfies (7.1).

Note that not all zircons are diamonds. For example, the two zircons in Figure 7.2 are not diamonds. Let us consider the poset on the right, the consideration about the left one being entirely similar.


Figure 7.4: zircon but not diamond

Let $M$ be the dashed special matching and $N$ be the dotted special matching. Then the pair $(M, N) \in S_{w} \times S_{w}$ does not satisfy the property of Definition 7.2.1. The reader can easily check this by noting that $\left|S_{w}\right|=6$, a special matching $F \in S_{w}$ being uniquely determinated by $F\left(a_{1}\right)$ and $F\left(a_{2}\right)$, with $2 \cdot 3$ possibilities. Another prove of that can be obtained, for example, by showing that

$$
\widetilde{R}_{M(c), M(w)}(q)+\chi(M(c) \triangleright c) q \widetilde{R}_{c, M(w)}(q)
$$

is not equal to

$$
\widetilde{R}_{N(c), N(w)}(q)+\chi(N(c) \triangleright c) q \widetilde{R}_{c, N(w)}(q)
$$

Now, $\widetilde{R}_{M(c), M(w)}(q)+\chi(M(c) \triangleright c) q \widetilde{R}_{c, M(w)}(q)=\widetilde{R}_{\hat{0}, a_{1}}=q^{3}$ by (7.6) since [0̂, $a_{1}$ ] is a 3 -krown, namely a Boolean algebra of length 3 .
On the contrary, by $(7.6), \widetilde{R}_{N(c), N(w)}(q)+\chi(N(c) \triangleright c) q \widetilde{R}_{c, N(w)}(q)=\widetilde{R}_{b_{3}, a_{1}}(q)+$ $\chi(N(c) \triangleright c) q \widetilde{R}_{c, a_{1}}(q)=q+q^{3}$.

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